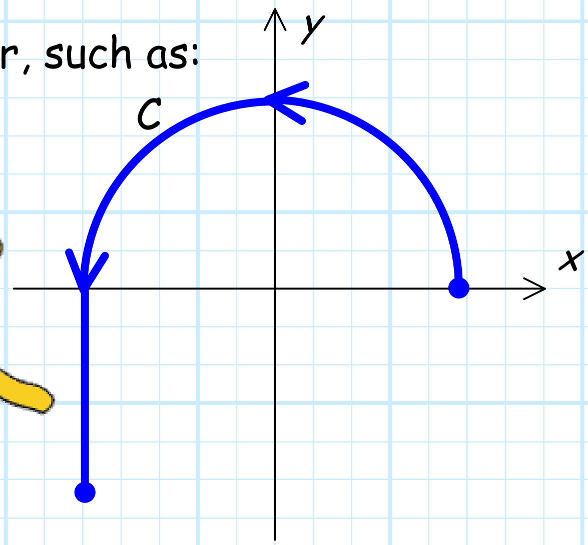
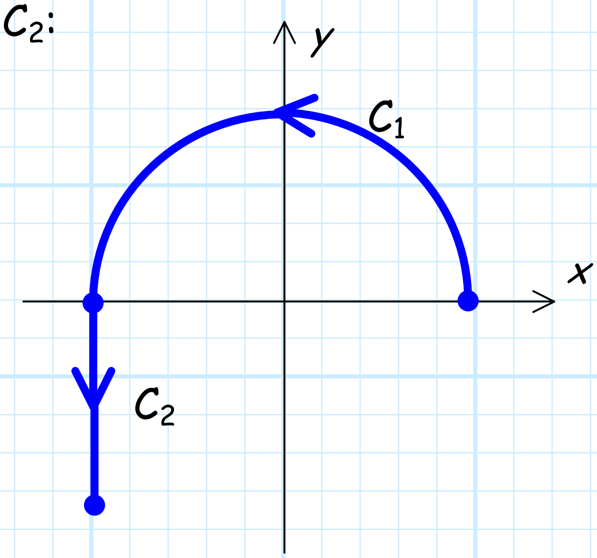
Line Integrals with Complex Contours

Consider a more **complex** contour, such as:



Q: *What's this flim-flam?! **This** contour can **neither** be expressed in terms of **single** coordinate inequality, **nor** with **single** differential line vector!*

A: True! But we can still **easily** evaluate a line integral over this contour C. The trick is to divide C into **two** contours, denoted as C_1 and C_2:



We can denote contour C as $C = C_1 + C_2$. It can be shown that:

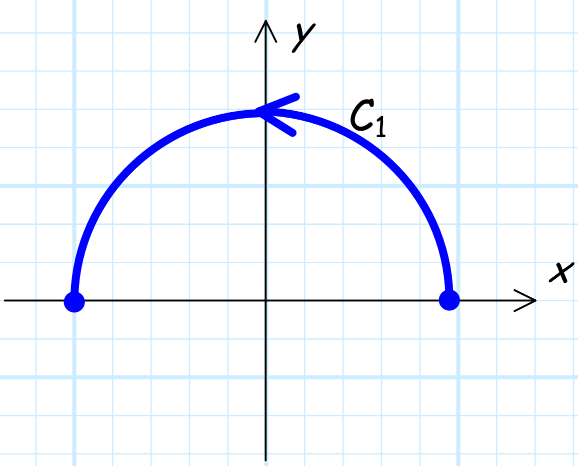
$$\int_C \mathbf{A}(\bar{r}_c)\cdot\overline{d\ell} = \int_{C_1} \mathbf{A}(\bar{r}_c)\cdot\overline{d\ell} + \int_{C_2} \mathbf{A}(\bar{r}_c)\cdot\overline{d\ell}$$

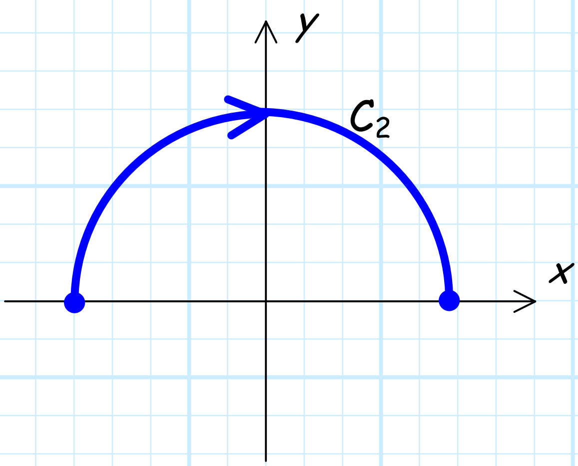
Note for the example given, we can evaluate the integral over both contour C_1 and contour C_2. The first is a **circular arc** around the z-axis, and the second is a **line segment** parallel to the y-axis.

Q: *Does the direction of the contour matter?*

A: **YES!** Every contour has a **starting** point and an **end** point. Integrating along the contour in the **opposite** direction will result in an **incorrect** answer!

For example, consider the two contours below:





In this case, the two contours are identical, with the **exception** of **direction**. In other words the beginning point of one is the end point of the other, and vice versa.

For this example, we would relate the two contours by saying:

$$C_1 = -C_2 \quad \text{and/or} \quad C_2 = -C_1$$

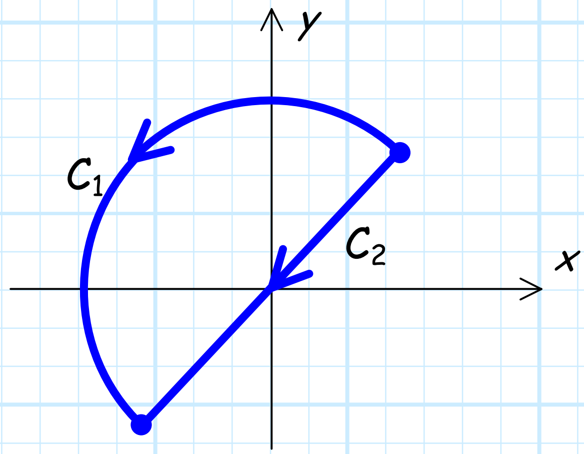
Just like vectors, the **negative** of a contour is an otherwise identical contour with opposite direction. We find that:

$$\int_{-C} \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell} = -\int_{C} \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell}$$

Q: *Does the* ***shape*** *of the contour* ***really*** *matter, or does the result of line integration only depend on the starting and end points ??*

A: Generally speaking, the shape of the contour **does** matter. Not only does the line integral depend on where we start and where we finish, it **also** depends on the path we take to get there!

For example, consider these two contours:



Generally speaking, we find that:

$$\int_{C_1} \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell} \neq \int_{C_2} \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell}$$

An **exception** to this is a **special** category of vector fields called **conservative** fields. For conservative fields, the contour path does **not** matter—the beginning and end points of the contour are **all** that are required to evaluate a line integral !