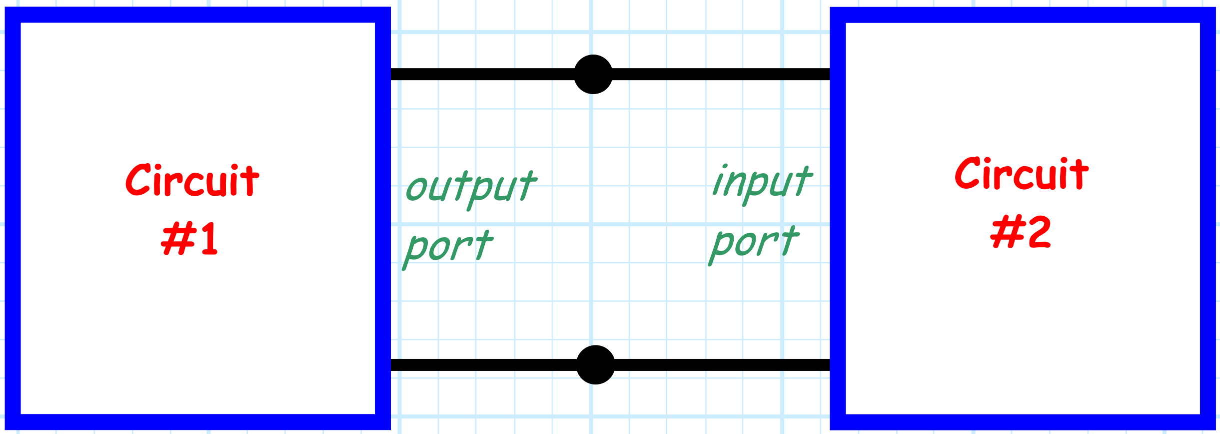
Impedance and Admittance Parameters

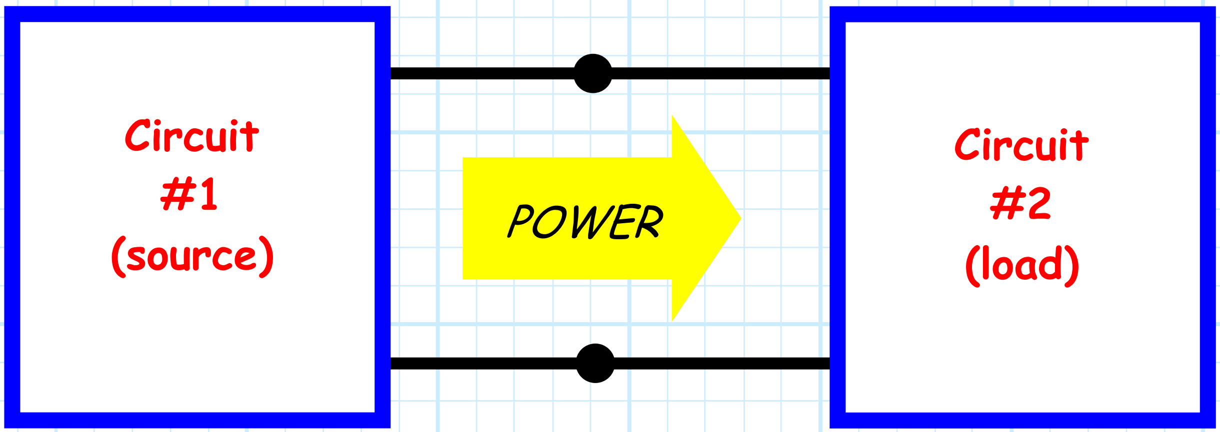
Say we wish to connect the **output** of one circuit to the **input** of another .



The terms "input" and "output" tells us that we wish for signal energy to flow **from** the output circuit **to** the input circuit.

Energy flows from source to load

In this case, the first circuit is the **source**, and the second circuit is the **load**.

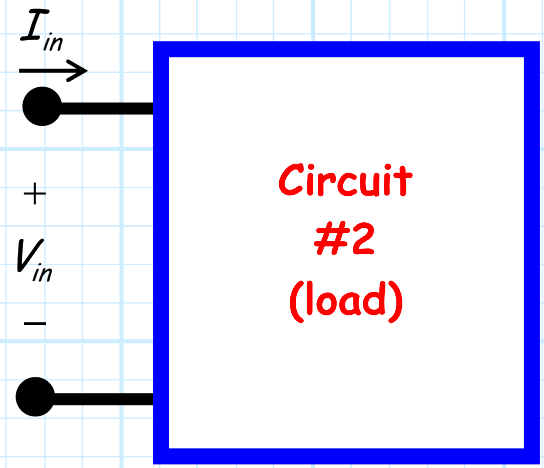


Each of these two circuits may be quite complex, but we can always simply this problem by using **equivalent circuits**.

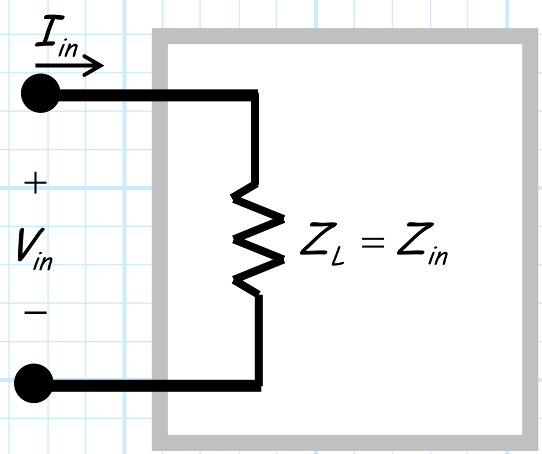
Load is the input impedance

For example, if we assume time-harmonic signals (i.e., eigen functions!), the load can be modeled as a simple lumped **impedance** , with a **complex** value equal to the input impedance of the circuit.

$$Z_{in} = \frac{V_{in}}{I_{in}}$$



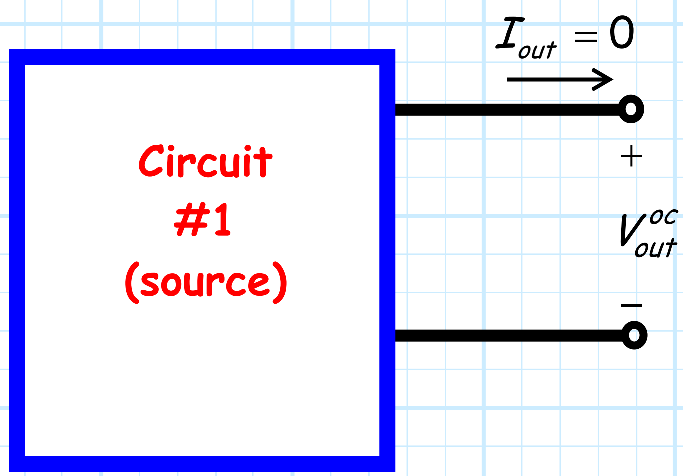
$$\boxed{V_{in} = Z_{in} I_{in}}$$



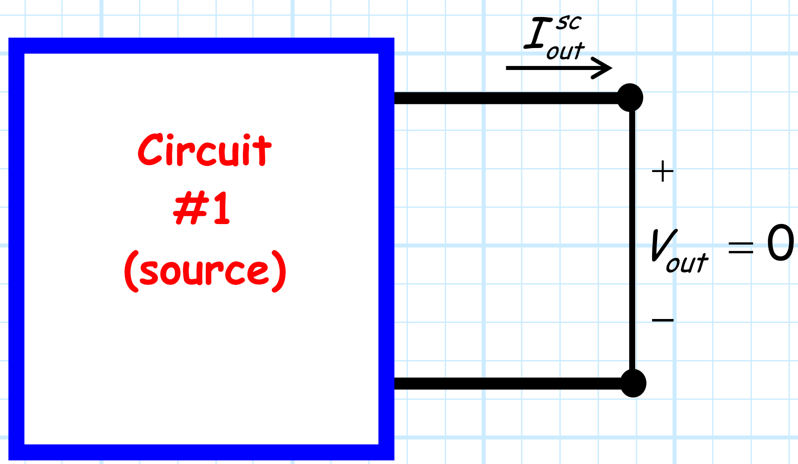
Equivalent Circuits

The source circuit can likewise be modeled using either a Thevenin's or Norton's equivalent.

This equivalent circuit can be determined by first evaluating (or measuring) the **open-circuit** output **voltage** V_{out}^{oc} :



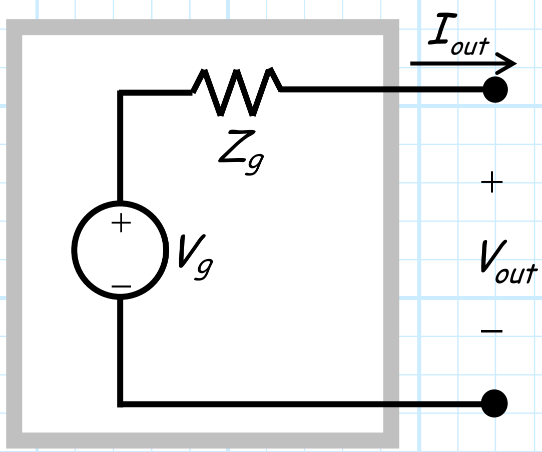
And likewise evaluating (or measuring) the **short-circuit** output **current** I_{out}^{sc} :



Thevenin's

From these two values (V_{out}^{oc} and I_{out}^{sc}) we can determine the **Thevenin's equivalent source**:

$$V_g = V_{out}^{oc} \qquad Z_g = \frac{V_{out}^{oc}}{I_{out}^{sc}}$$



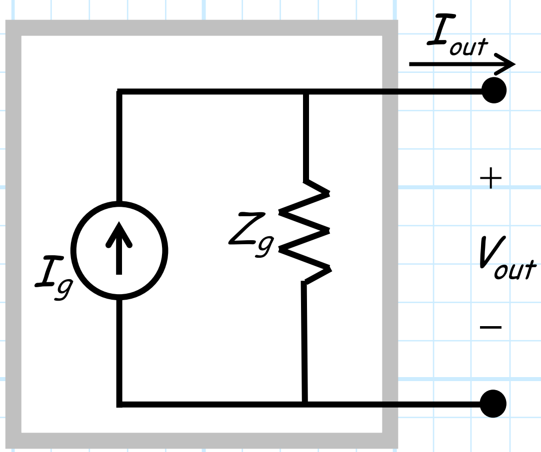
$$V_{out} = V_g - Z_g I_{out}$$

$$I_{out} = \frac{V_g - V_{out}}{Z_g}$$

Norton's

Or, we could use a **Norton's** equivalent circuit:

$$I_g = I_{out}^{sc} \qquad Z_g = \frac{V_{out}^{oc}}{I_{out}^{sc}}$$

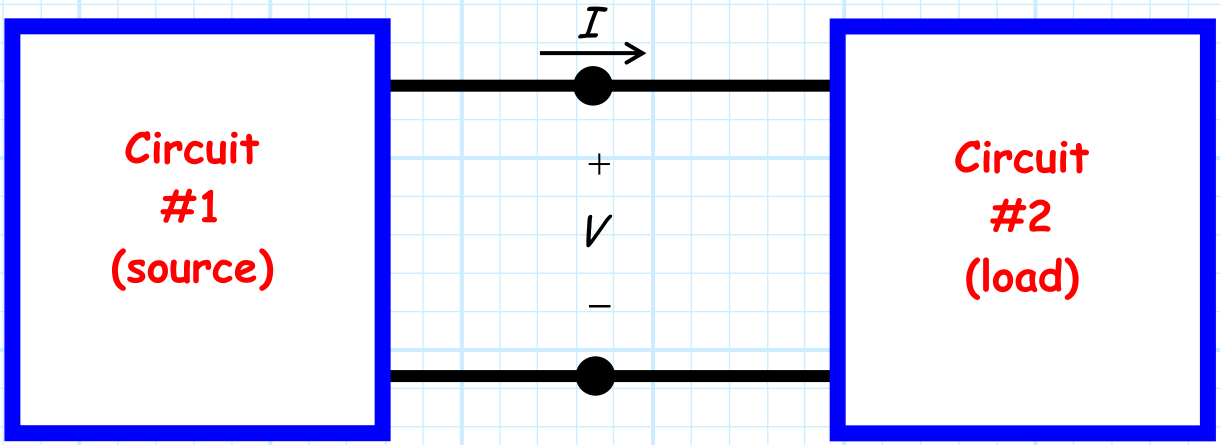


$$I_{out} = I_g - V_{out}/Z_g$$

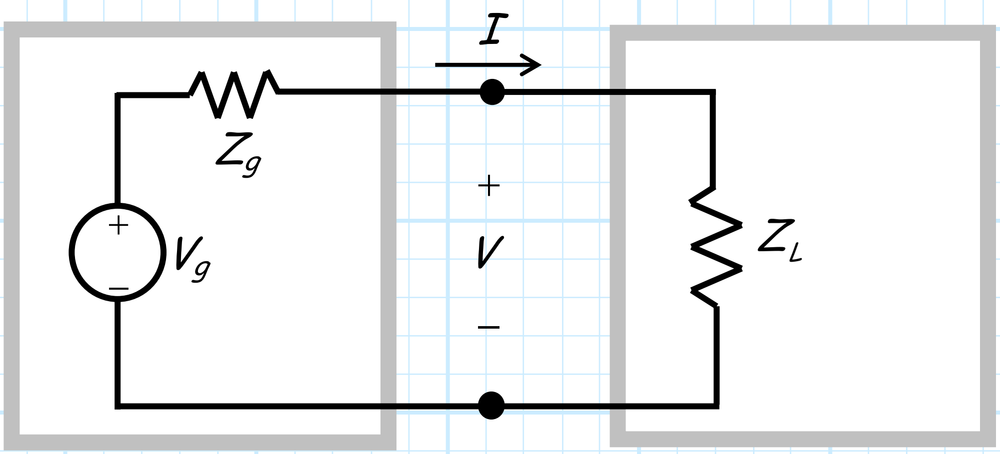
$$V_{out} = \left(I_g - I_{out}\right) Z_g$$

Circuit Model

Thus, the entire circuit:



Can be modeled with equivalent circuits as:



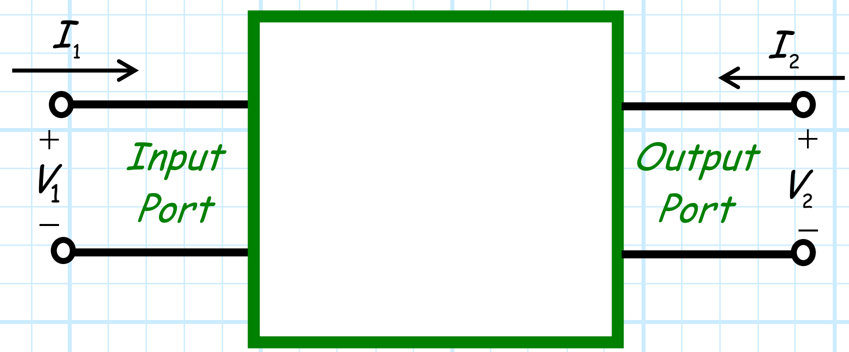
Please note again that we have assumed a **time harmonic** source, such that all the values in the circuit above (V_g, Z_g, I, V, Z_L) are **complex** (i.e., they have a **magnitude** and **phase**).

Two-Port circuits

Q: *But, circuits like filters and amplifiers are two-port devices, they have both an input and an output. How do we characterize a two-port device?*

A: Indeed, many important components are **two-port** circuits.

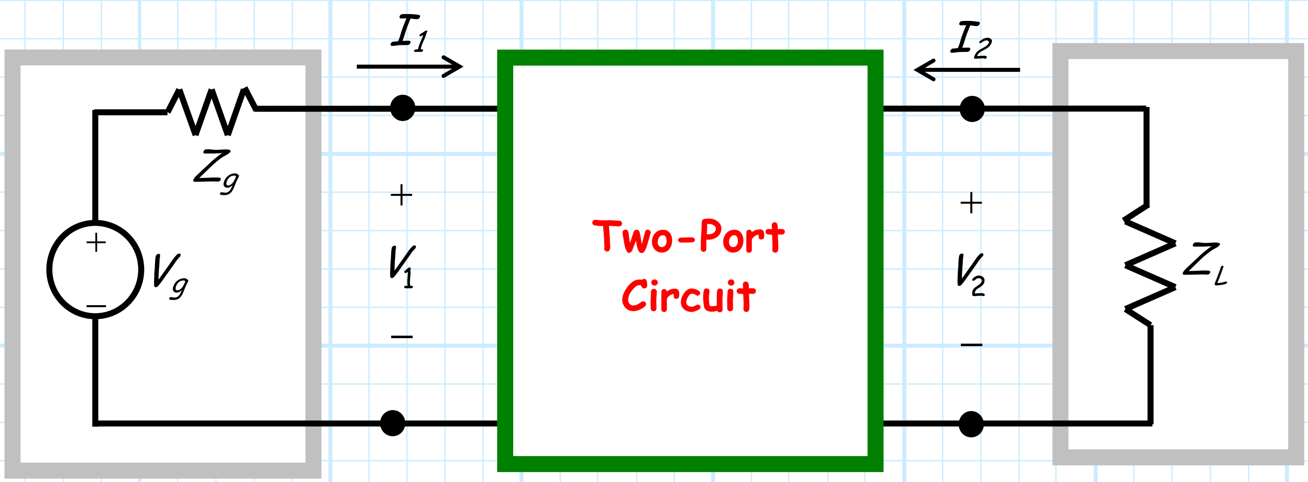
For these devices, the signal power **enters** one port (i.e., the input) and **exits** the other (the output).



Between source and load

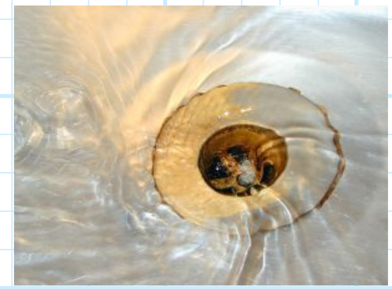
These two-port circuits typically do something to **alter** the signal as it passes from input to output (e.g., filters it, amplifies it, attenuates it).

We can thus assume that a **source** is connected to the **input** port, and that a **load** is connected to the **output** port.



How to characterize?

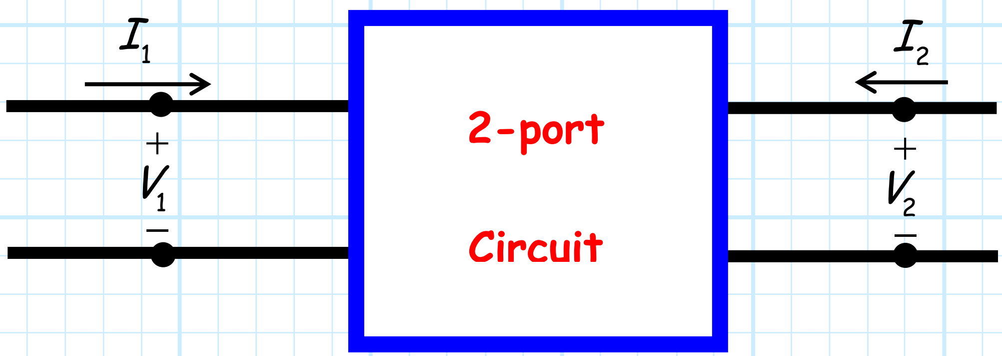
Again, the source circuit may be **quite complex**, consisting of many components. However, at least one of these components must be a **source** of energy.



Likewise, the load circuit might be **quite complex**, consisting of many components. However, at least one of these components must be a **sink** of energy.

Q: *But what about the **two-port circuit** in the middle? How do we characterize it?*

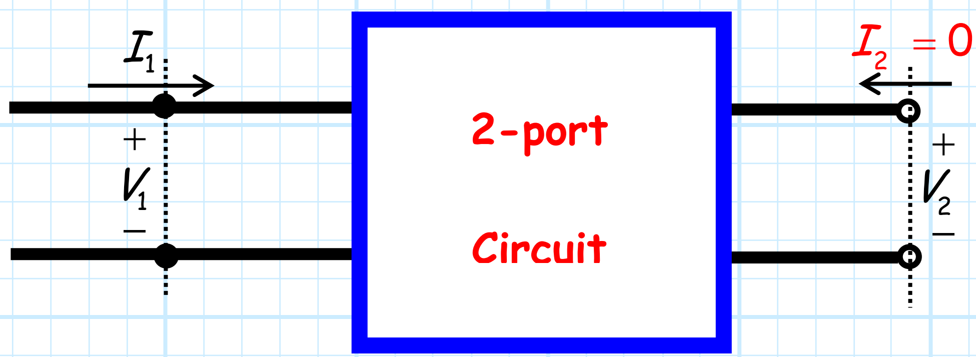
A: A linear two-port circuit is fully characterized by just four **impedance parameters**!



Do this little experiment

Note that inside the "blue box" there could be anything from a very **simple** linear circuit to a very large and **complex** linear system.

Now, say there exists a non-zero current at input **port 1** (i.e., $I_1 \neq 0$), while the current at **port 2** is known to be **zero** (i.e., $I_2 = 0$).



Say we measure/determine the **current** at port **1** (i.e., determine I_1), and we then measure/determine the **voltage** at the port **2** plane (i.e., determine V_2).

Impedance parameters

The complex ratio between V_2 and I_1 is know as the **trans-impedance parameter** Z_{21}:

$$Z_{21}(\omega) = \frac{V_2(\omega)}{I_1(\omega)}$$

Note this trans-impedance parameter is the Eigen value of the linear operator relating current $i_1(t)$ to voltage $v_2(t)$:

$$v_2(t) = \mathcal{L}\{i_1(t)\} \quad \rightarrow \quad V_2(\omega) = G_{21}(\omega)\, I_1(\omega)$$

Thus:

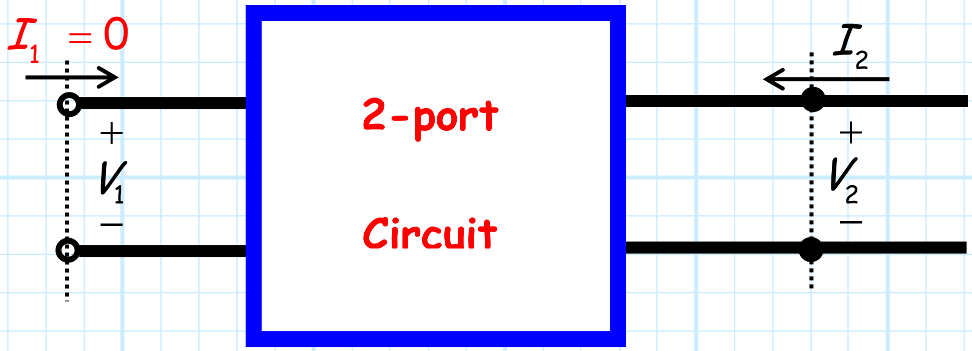
$$G_{21}(\omega) = Z_{21}(\omega)$$

Likewise, the complex ratio between V_1 and I_1 is the **trans-impedance parameter** Z_{11}:

$$Z_{11}(\omega) = \frac{V_1(\omega)}{I_1(\omega)}$$

A second experiment

Now consider the opposite situation, where there exists a non-zero current at **port 2** (i.e., $I_2 \neq 0$), while the current at port 1 is known to be **zero** (i.e., $I_2 = 0$).



The result is two more impedance parameters:

$$Z_{12}(\omega) = \frac{V_1(\omega)}{I_2(\omega)} \qquad Z_{22}(\omega) = \frac{V_2(\omega)}{I_2(\omega)}$$

Thus, more **generally**, the ratio of the current into port n and the voltage at port m is:

$$Z_{mn} = \frac{V_m}{I_n} \qquad \text{(given that } I_k = 0 \text{ for } k \neq n\text{)}$$

Open circuits enforce I=0

Q: *But how do we ensure that* ***one*** *port current is zero ?*

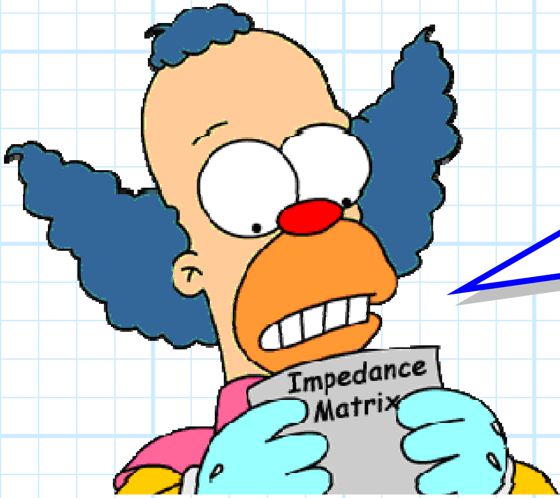
A: Place an **open** circuit **at** that port!

Placing an **open** at a port (and it must be **at** the port!) **enforces** the condition that $I = 0$.

Now, we can thus **equivalently** state the definition of trans-impedance as:

$$Z_{mn} = \frac{V_m}{I_n} \qquad \text{(given that port } k \neq n \text{ is } \textbf{open-circuited}\text{)}$$

What's the point?



*Q: As impossible as it sounds, this handout is even more **pointless** than all your previous efforts. **Why** are we studying this? After all, what is the likelihood that a device will have an **open** circuit on one of its ports?!*

A: OK, say that **neither** port is **open-circuited**, such that we have currents **simultaneously** on **both** of the two ports of our device.

Since the device is **linear**, the voltage at **one** port is due to **both** port currents.

This voltage is simply the coherent **sum** of the voltage at that port due to **each** of the two currents!

Specifically, the voltage at each port can is:

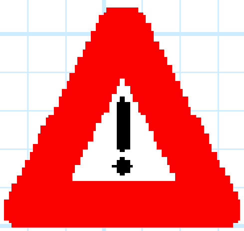
$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$

$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

They're a function of frequency!

Thus, these four impedance parameters **completely characterizes** a linear, *2*-port device.

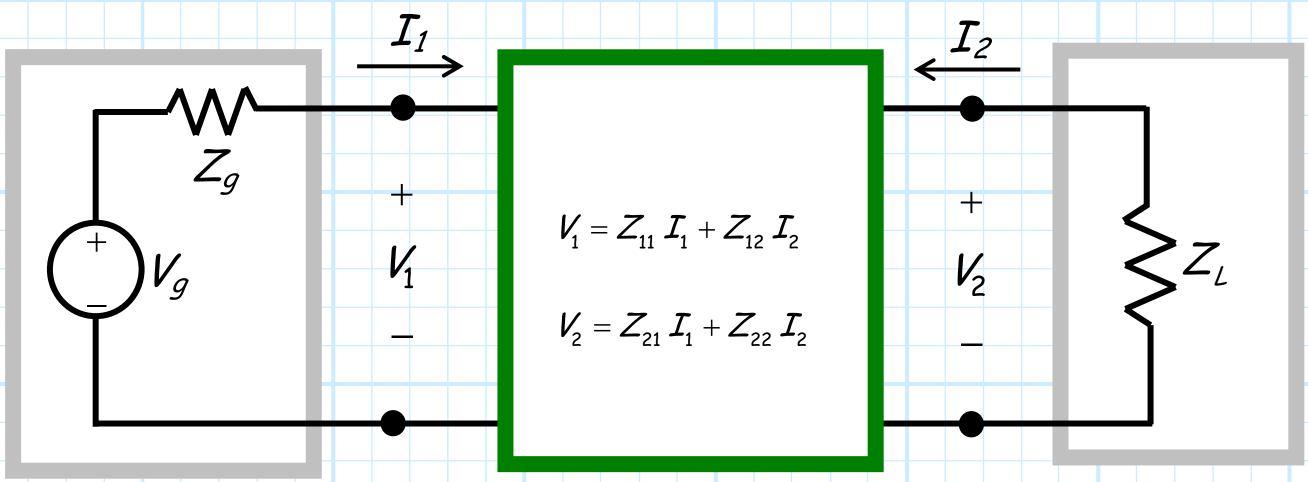
Effectively, these impedance parameters describes a 2-port device the way that Z_L describes a single-port device (e.g., a load)!



But **beware**! The values of the impedance matrix for a particular device or circuit, just like Z_L, are **frequency dependent**!

A complete equivalent circuit

Now, we can use our **equivalent circuits** to model this system:



Note in this circuit there are **4 unknown values**—two voltages (V_1 and V_2), and two currents (I_1 and I_2).

→ Our job is to **determine** these 4 unknown values!

Let's do some algebra!

Let's begin by looking at the source, we can determine from **KVL** that:

$$V_g - Z_g I_1 = V_1$$

And so with a bit of **algebra**:

$$I_1 = \frac{V_g - V_1}{Z_g}$$ (←look, **Ohm's Law**!)

Now let's look at our two-port circuit. If we know the impedance matrix (i.e., all **four trans-impedance** parameters), then:

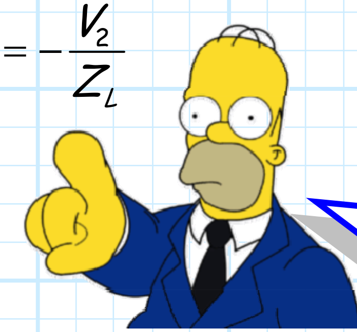
$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

Watch the minus sign!

Finally, for the load:

$$I_2 = -\frac{V_2}{Z_L}$$



Q: *Are you sure this is correct? I don't recall there being a* ***minus sign*** *in Ohm's Law.*

A: Be **very careful** with the notation.

Current I_2 is defined as positive when it is flowing into the two port circuit. This is the notation required for the impedance matrix.

Thus, positive current I_2 is flowing out of the load impedance—the opposite convention to Ohm's Law.

This is why the **minus sign is required**.

A very good thing

Now let's **take stock** of our results. Notice that we have compiled **four** independent equations, involving our **four** unknown values:

$$I_1 = \frac{V_g - V_1}{Z_g}$$

Q: *Four equations and four unknowns! That sounds like a **very good thing**!*

$$I_2 = -\frac{V_2}{Z_L}$$

$$V_1 = Z_{11}\, I_1 + Z_{12}\, I_2$$

$$V_2 = Z_{21}\, I_1 + Z_{22}\, I_2$$

A: It **is**! We can apply a bit of **algebra** and solve for the unknown currents and voltages:

$$I_1 = V_g \frac{Z_{22} + Z_L}{\left(Z_{11} + Z_g\right)\left(Z_{22} + Z_L\right) - Z_{12} Z_{21}} \qquad I_2 = -V_g \frac{Z_{21}}{\left(Z_{11} + Z_g\right)\left(Z_{22} + Z_L\right) - Z_{12} Z_{21}}$$

$$V_1 = V_g \frac{Z_{11}\left(Z_{22} + Z_L\right) - Z_{12} Z_{21}}{\left(Z_{11} + Z_g\right)\left(Z_{22} + Z_L\right) - Z_{12} Z_{21}} \qquad V_2 = V_g \frac{Z_L\, Z_{21}}{\left(Z_{11} + Z_g\right)\left(Z_{22} + Z_L\right) - Z_{12} Z_{21}}$$

Admittance Parameters

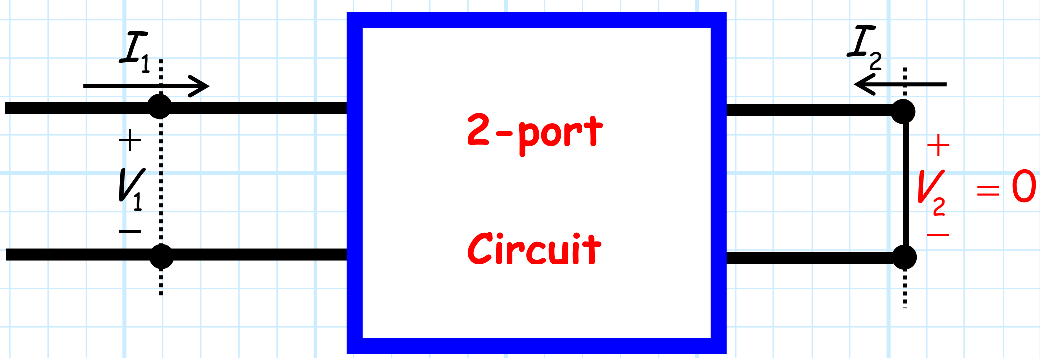
Q: *Are impedance parameters the **only** way to characterize a 2-port linear circuit?*

A: Hardly! Another method uses **admittance parameters**.

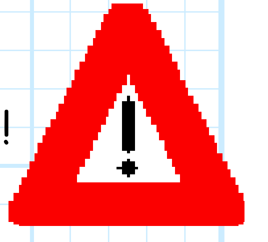
The elements of the Admittance Matrix are the **trans-admittance** parameters Y_{mn}, defined as:

$$Y_{mn} = \frac{I_m}{V_n} \qquad \text{(given that } V_k = 0 \text{ for } k \neq n\text{)}$$

Note here that the **voltage** at one port **must** be equal to **zero**. We can ensure that by simply placing a **short circuit** at the zero-voltage port!



Note that $Y_{mn} \neq 1/Z_{mn}$!



Short circuits enforce V=0

Now, we can **equivalently** state the definition of trans-admittance as:

$$Y_{mn} = \frac{V_m}{I_n} \qquad \text{(given that all ports } k \neq n \text{ are } \textbf{short - circuited}\text{)}$$

Just as with the trans-impedance values, we can use the trans-admittance values to evaluate general circuit problems, where **none** of the ports have zero voltage.

Since the device is **linear**, the current at any **one** port due to **all** the port currents is simply the coherent **sum** of the currents at that port due to **each** of the port voltages!

$$I_1 = Y_{11}\, V_1 + Y_{12}\, V_2$$

$$I_2 = Y_{21}\, V_1 + Y_{22}\, V_2$$