2. Microwave School

We design radio systems using **RF/microwave** components.

Q: *Why don't we use the "usual" circuit components (e.g., resistors, capacitors, op-amps, transistors) ??*

A: We **do** use these! But we require new devices because:

1. Our circuits are generally > λ in **size** !

2. We require **new** functions that "non-RF" devices cannot provide.

A. *Transmission Line Theory*

The most important fact about microwave devices is that they are connected together using transmission lines.

Q: *So just what is a **transmission line?***

A: A passive, linear, two port device that allows bounded E. M. energy to flow from one device to another.

→ Sort of an "electromagnetic pipe" !

Q: *Oh, so it's simply a conducting **wire**, right?*

A: NO! At high frequencies, things get much more complicated!

HO: The Telegraphers Equations

HO: Time-Harmonic Solutions for Linear Circuits

Q: *So, what complex functions I(z) and V(z)* ***do*** *satisfy both telegrapher equations?*

A: The solutions to the transmission line **wave equations**!

HO: The Transmission Line Wave Equations

Q: *Are the solutions for I(z) and V(z) completely* ***independent****, or are they* ***related*** *in any way ?*

A: The two solutions are related by the transmission line **characteristic impedance**.

HO: The Transmission Line Characteristic Impedance

Q: *So what is the significance of the constant* β*? What does it tell us?*

A: It describes the **propagation** of each **wave** along the transmission line.

HO: The Propagation Constant

Q: *Is **characteristic** impedance Z_0 the same as the concept of impedance I learned about in **circuits** class?*

A: **NO!** The Z_0 is a **wave** impedance. However, we can also define **line impedance**, which **is** the same as that used in circuits.

HO: Line Impedance

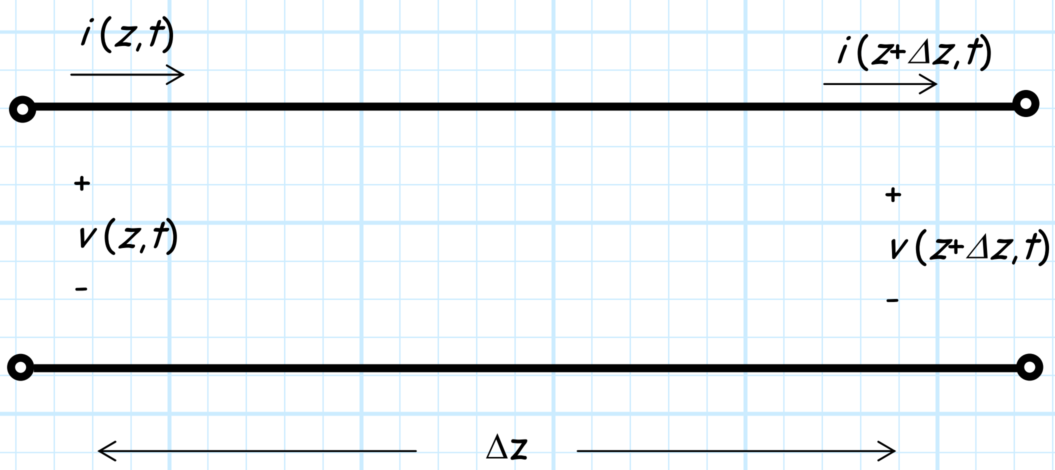
Q: *These **wave** functions $V^{+}(z)$ and $V^{-}(z)$ seem to be important. How are **they** related?*

A: They are in fact **very** important! They are related by a function called the **reflection coefficient**.

HO: The Reflection Coefficient

The Telegrapher Equations

Consider a section of "wire":



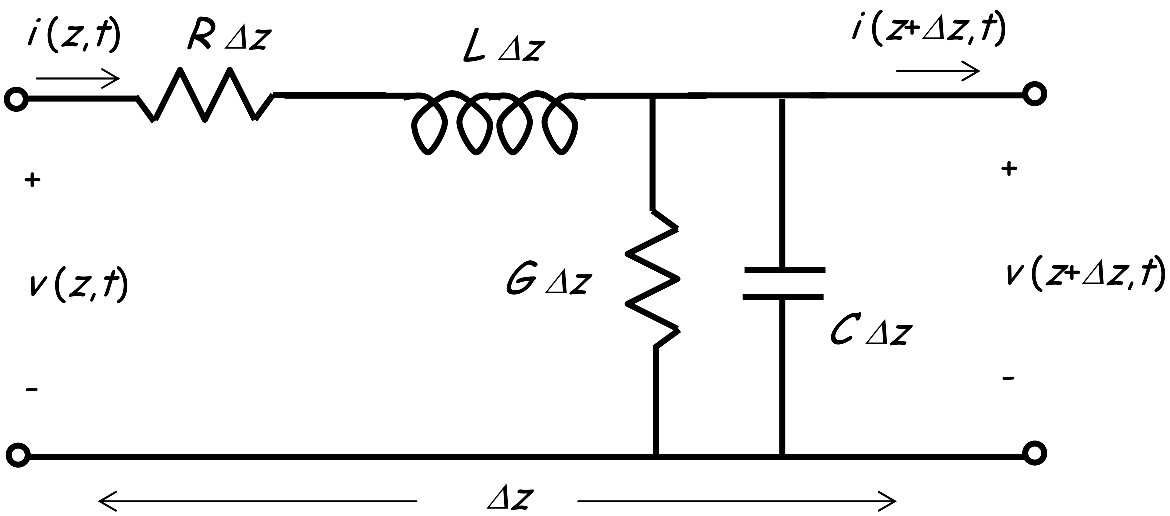
Q: *Huh ?! Current i and voltage v are a function of* ***position*** *z ??*
Shouldn't $i(z,t) = i(z+\Delta z,t)$ *and* $v(z,t) = v(z+\Delta z,t)$ *?*

A: **NO !** Because a wire is never a **perfect** conductor.

A "wire" will have:

1) Inductance
2) Resistance
3) Capacitance
4) Conductance

i.e.,



Where:

R = resistance/unit length
L = inductance/unit length
C = capacitance/unit length
G = conductance/unit length

∴ resistance of wire length Δz is $R\Delta z$.

Using KVL, we find:

$$v(z+\Delta z,t)-v(z,t)=-R\Delta z\, i(z,t)-L\Delta z\frac{\partial i(z,t)}{\partial t}$$

and from KCL:

$$i(z+\Delta z,t)-i(z,t)=-G\Delta z\, v(z,t)-C\Delta z\frac{\partial v(z,t)}{\partial t}$$

Dividing the first equation by Δz, and then taking the limit as $\Delta z \rightarrow 0$:

$$\lim_{\Delta z \rightarrow 0} \frac{v(z+\Delta z,t)-v(z,t)}{\Delta z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

which, by definition of the derivative, becomes:

$$\frac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

Similarly, the KCL equation becomes:

$$\frac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C\frac{\partial v(z,t)}{\partial t}$$

These equations are known as the **telegrapher's equations** !

$$\frac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

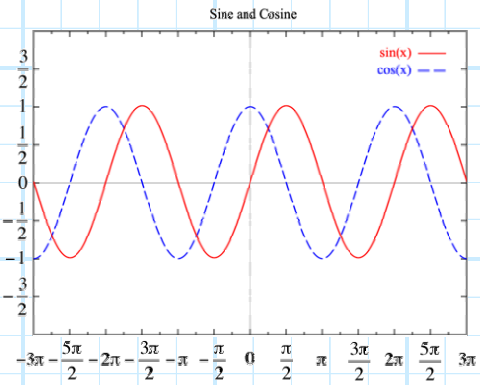
$$\frac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C\frac{\partial v(z,t)}{\partial t}$$

Time-Harmonic Solutions for Linear Circuits

There are an unaccountably **infinite** number of solutions $v(z,t)$ and $i(z,t)$ for the telegrapher's equations! However, we can simplify the problem by assuming that the function of time is **time harmonic** (i.e., sinusoidal), oscillating at some radial **frequency** ω (e.g., $\cos \omega t$).

Q: *Why on earth would we assume a **sinusoidal** function of time? Why not a **square wave**, or **triangle wave**, or a "sawtooth" function?*

A: We assume **sinusoids** because they have a **very** special property!



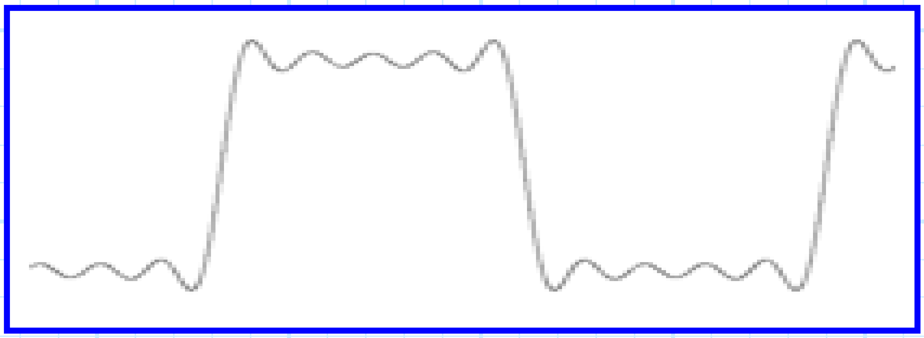
Sinusoidal time functions—and **only** a sinusoidal time functions—are the **eigen functions** of **linear, time-invariant** systems.

Q: *???*

A: If a sinusoidal voltage source with frequency ω is used to excite a linear, time-invariant circuit (and a transmission line is **both** linear **and** time invariant!), then the voltage at each and **every** point with the circuit will likewise vary sinusoidally—at the same frequency ω!

Q: *So what? Isn't that **obvious**?*

A: Not at all! If you were to excite a linear circuit with a **square wave**, or **triangle wave**, or **sawtooth**, you would find that—generally speaking—**nowhere** else in the circuit is the voltage a perfect square wave, triangle wave, or sawtooth. The linear circuit will effectively **distort** the input signal into something **else**!



Q: *Into what function will the input signal be distorted?*

A: It depends—both on the original form of the **input signal**, and the parameters of the **linear circuit**. At **different** points within the circuit we will discover **different** functions of time—**unless**, of course, we use a **sinusoidal** input. Again, for a sinusoidal excitation, we find at **every** point within circuit an **undistorted** sinusoidal function!

Q: *So, the sinusoidal function at every point in the circuit is **exactly** the same as the input sinusoid?*

A: Not quite **exactly** the same. Although at every point within the circuit the voltage will be precisely sinusoidal (with frequency ω), the **magnitude** and **relative phase** of the sinusoid will generally be different at each and every point within the circuit.

Thus, the voltage along a transmission line—when excited by a sinusoidal source—**must** have the form:

$$v(z,t) = v(z)\cos\left(\omega t + \varphi(z)\right)$$

Thus, at some arbitrary location z along the transmission line, we **must** find a time-harmonic oscillation of **magnitude** $v(z)$ and **relative phase** $\varphi(z)$.

Now, consider Euler's equation, which states:

$$e^{j\psi} = \cos\psi + j\sin\psi$$

Thus, it is apparent that:

$$Re\left\{e^{j\psi}\right\} = \cos\psi$$

and so we conclude that the voltage on a transmission line can be expressed as:

$$\begin{aligned} v(z,t) &= v(z)\cos\left(\omega t + \varphi(z)\right) \\ &= Re\left\{v(z)\,e^{j(\omega t + \varphi(z))}\right\} \\ &= Re\left\{v(z)\,e^{+j\varphi(z)}e^{j\omega t}\right\} \end{aligned}$$

Thus, we can specify the time-harmonic voltage at each an every location z along a transmission line with the **complex** function $V(z)$:

$$V(z) = v(z)\, e^{-j\varphi(z)}$$

where the **magnitude** of the complex function is the **magnitude** of the sinusoid:

$$v(z) = |V(z)|$$

and the phase of the complex function is the relative phase of the sinusoid :

$$\varphi(z) = arg\{V(z)\}$$

Q: *Hey wait a minute! What happened to the time-harmonic function $e^{j\omega t}$??*

A: There really is no reason to **explicitly** write the complex function $e^{j\omega t}$, since we know in fact (being the eigen function of linear systems and all) that if this is the time function at any **one** location (such as qt the excitation source) then this must be time function at **all** transmission line locations z!

The only **unknown** is the **complex** function $V(z)$. Once we determine $V(z)$, we can always (if we so desire) "recover" the **real** function $v(z,t)$ as:

$$v(z,t) = Re\{V(z)\, e^{j\omega t}\}$$

Thus, if we assume a **time-harmonic source**, finding the transmission line solution $v(z,t)$ reduces to solving for the **complex function** $V(z)$.

The Transmission Line Wave Equation

Let's assume that $v(z,t)$ and $i(z,t)$ each have the time-harmonic form:

$$v(z,t) = \text{Re}\{V(z)e^{j\omega t}\} \quad \text{and} \quad i(z,t) = \text{Re}\{I(z)e^{j\omega t}\}$$

The time-derivative of these functions are:

$$\frac{\partial v(z,t)}{\partial t} = \text{Re}\left\{V(z)\frac{\partial e^{j\omega t}}{\partial t}\right\} = \text{Re}\{j\omega V(z)e^{j\omega t}\}$$

$$\frac{\partial i(z,t)}{\partial t} = \text{Re}\left\{I(z)\frac{\partial e^{j\omega t}}{\partial t}\right\} = \text{Re}\{j\omega I(z)e^{j\omega t}\}$$

The telegrapher's equations thus become:

$$Re\left\{\frac{\partial V(z)}{\partial z}e^{j\omega t}\right\} = Re\{-(R + j\omega L)I(z)\,e^{j\omega t}\}$$

$$Re\left\{\frac{\partial I(z)}{\partial z}e^{j\omega t}\right\} = Re\{-(G + j\omega C)V(z)\,e^{j\omega t}\}$$

And then simplifying, we have the **complex** form of **telegrapher's equations**:

$$\frac{\partial V(z)}{\partial z} = -(R + j\omega L)\, I(z)$$

$$\frac{\partial I(z)}{\partial z} = -(G + j\omega C)\, V(z)$$

Note that these complex differential equations are **not** a function of time t !

* The functions *I(z)* and *V(z)* are **complex**, where the **magnitude** and **phase** of the complex functions describe the **magnitude** and **phase** of the sinusoidal time function $e^{j\omega t}$.

* Thus, *I(z)* and *V(z)* describe the current and voltage along the transmission line, as a function as position *z*.

* **Remember**, not just **any** function *I(z)* and *V(z)* can exist on a transmission line, but rather **only** those functions that satisfy the **telegraphers equations**.

Our task, therefore, is to **solve** the telegrapher equations and find **all** solutions $I(z)$ and $V(z)$!

Q: *So, what functions I (z) and V (z)* ***do*** *satisfy both telegrapher's equations??*

A: To make this easier, we will combine the telegrapher equations to form **one** differential equation for ***V (z)*** and **another** for ***I(z)***.

First, take the **derivative** with respect to z of the **first** telegrapher equation:

$$\frac{\partial}{\partial z}\left\{\frac{\partial V(z)}{\partial z} = -(R + j\omega L)\, I(z)\right\}$$

$$= \frac{\partial^2 V(z)}{\partial z^2} = -(R + j\omega L)\frac{\partial I(z)}{\partial z}$$

Note that the **second** telegrapher equation expresses the derivative of $I(z)$ in terms of $V(z)$:

$$\frac{\partial I(z)}{\partial z} = -(G + j\omega C)\, V(z)$$

Combining these two equations, we get an equation involving $V(z)$ **only**:

$$\frac{\partial^2 V(z)}{\partial z^2} = (R + j\omega L)(G + j\omega C)\, V(z)$$

Now, we find at high frequencies that:

$$R \ll j\omega L \quad \text{and} \quad G \ll j\omega C$$

and so we can **approximate** the differential equation as:

$$\frac{\partial^2 V(z)}{\partial z^2} = (j\omega L)(j\omega C)V(z) = \omega^2 LC\, V(z) = \beta^2\, V(z)$$

where it is apparent that:

$$\beta^2 \doteq \omega^2 LC$$

In a **similar** manner (i.e., begin by taking the derivative of the **second** telegrapher equation), we can derive the differential equation:

$$\frac{\partial^2 I(z)}{\partial z} = \beta^2\, I(z)$$

We have **decoupled** the telegrapher's equations, such that we now have **two** equations involving **one** function only:

$$\frac{\partial^2 V(z)}{\partial z} = \beta^2 V(z)$$

$$\frac{\partial^2 I(z)}{\partial z} = \beta^2\, I(z)$$

These are known as the **transmission line wave equations**.

Note only **special** functions satisfy these equations: if we take the double derivative of the function, the result is the **original function** (to within a constant)!

Q: *Yeah right! Every function that I know is **changed** after a double differentiation. What kind of "magical" function could possibly satisfy **this** differential equation?*

A: Such functions **do** exist !

For example, the functions $V(z) = e^{-j\beta z}$ and $V(z) = e^{+j\beta z}$ each satisfy this transmission line wave equation (**insert** these into the differential equation and see for **yourself**!).

Likewise, since the transmission line wave equation is a **linear** differential equation, a weighted **superposition** of the two solutions is **also a solution** (again, **insert** this solution to and see for **yourself**!):

$$V(z) = V_0^+ \, e^{-j\beta z} + V_0^- \, e^{+j\beta z}$$

In fact, it turns out that **any** and **all** possible solutions to the differential equations can be expressed in **this** simple form!

Therefore, the **general** solution to these wave equations (and thus the telegrapher equations) are:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

$$I(z) = I_0^+ e^{-j\beta z} + I_0^- e^{+j\beta z}$$

where V_0^+, V_0^-, I_0^+, and I_0^- are **complex constants**.

→ It is **unfathomably** important that **you** understand what this result means!

It means that the functions $V(z)$ and $I(z)$, describing the current and voltage at **all** points z along a transmission line, can **always** be **completely** specified with just **four complex constants** $(V_0^+, V_0^-, I_0^+, I_0^-)$!!

We can **alternatively** write these solutions as:

$$V(z) = V^+(z) + V^-(z)$$

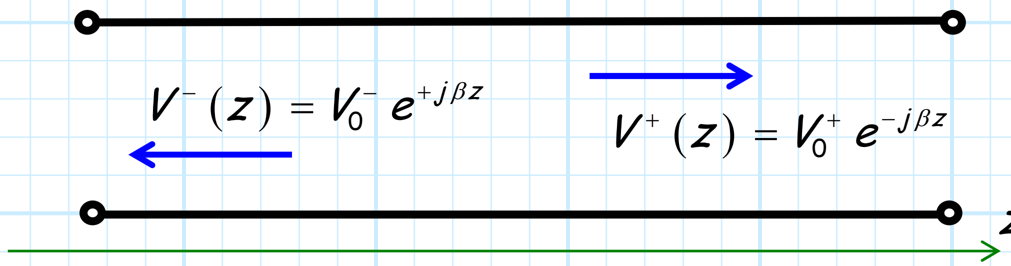
$$I(z) = I^+(z) + I^-(z)$$

where:

$$V^+(z) \doteq V_0^+ e^{-j\beta z} \qquad V^-(z) \doteq V_0^- e^{+j\beta z}$$

$$I^+(z) \doteq I_0^+ e^{-j\beta z} \qquad I^-(z) \doteq I_0^- e^{+j\beta z}$$

The two terms in each solution describe **two waves** propagating in the transmission line, **one** wave ($V^+(z)$ or $I^+(z)$) propagating in one direction (**+*z***) and the **other** wave ($V^-(z)$ or $I^-(z)$) propagating in the **opposite** direction (**-*z***).



Q: *So just what* ***are*** *the complex values V_0^+, V_0^-, I_0^+, I_0^- ?*

A: Consider the wave solutions at **one** specific point on the transmission line—the point $z = 0$. For example, we find that:

$$\begin{aligned} V^+(z=0) &= V_0^+ e^{-j\beta(z=0)} \\ &= V_0^+ e^{-(0)} \\ &= V_0^+ (1) \\ &= V_0^+ \end{aligned}$$

In other words, V_0^+ is simply the **complex** value of the wave function $V^+(z)$ at the point z =0 on the transmission line!

Likewise, we find:

$$V_0^- = V^-(z=0)$$

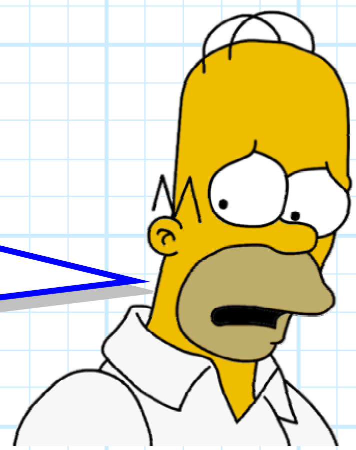
$$I_0^+ = I^+(z=0)$$

$$I_0^- = I^-(z=0)$$

Again, the four complex values V_0^+, I_0^+, V_0^-, I_0^- are **all** that is needed to determine the voltage and current at any and all points on the transmission line.

More specifically, **each** of these four complex constants completely specifies **one** of the four transmission line wave functions $V^+(z)$, $I^+(z)$, $V^-(z)$, $I^-(z)$.

Q: *But what **determines** these wave functions? How do we **find** the values of constants* V_0^+, I_0^+, V_0^-, I_0^-?



A: As you might expect, the voltage and current on a transmission line is determined by the devices **attached** to it on either end (e.g., active sources and/or passive loads)!

The precise values of V_0^+, I_0^+, V_0^-, I_0^- are therefore determined by satisfying the **boundary conditions** applied at **each end** of the transmission line—much more on this **later**!

The Characteristic Impedance of a Transmission Line

So, from the telegrapher's differential equations, we know that the complex current $I(z)$ and voltage $V(z)$ **must** have the form:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

$$I(z) = I_0^+ e^{-j\beta z} + I_0^- e^{+j\beta z}$$

Let's insert the expression for $V(z)$ into the first telegrapher's equation, and **see what happens**!

$$\frac{dV(z)}{dz} = -j\beta V_0^+ e^{-j\beta z} + j\beta V_0^- e^{+j\beta z} = -j\omega L\, I(z)$$

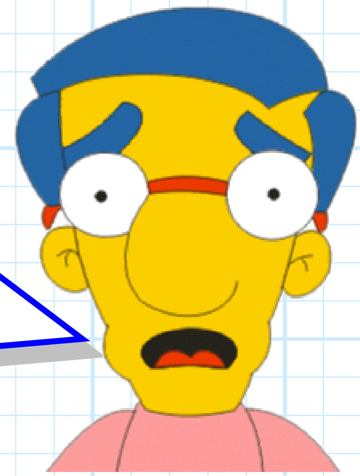
Therefore, rearranging, $I(z)$ must be:

$$I(z) = \frac{\beta}{\omega L}\left(V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}\right)$$

Q: ***But wait*** *! I thought we already knew current I(z). Isn't it:*

$$I(z) = I_0^+ e^{-j\beta z} + I_0^- e^{+j\beta z} \quad ??$$

*How can **both** expressions for I(z) be true??*



A: Easy ! Both expressions for current are **equal** to each other.

$$I(z) = I_0^+ e^{-j\beta z} + I_0^- e^{+j\beta z} = \frac{\beta}{\omega L}\left(V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}\right)$$

For the above equation to be true for **all** z, I_0 and V_0 must be related as:

$$I_0^+ e^{-\gamma z} = \left(\frac{\beta}{\omega L}\right) V_0^+ e^{-\gamma z} \qquad \text{and} \qquad I_0^- e^{+\gamma z} = \left(\frac{-\beta}{\omega L}\right) V_0^- e^{+\gamma z}$$

Or—recalling that $V_0^+ e^{-j\beta z} = V^+(z)$ (etc.)—we can express this in terms of the **two propagating waves**:

$$I^+(z) = \left(\frac{\beta}{\omega L}\right) V^+(z) \qquad \text{and} \qquad I^-(z) = \left(\frac{-\beta}{\omega L}\right) V^-(z)$$

Now, we note that since:

$$\beta = \omega\sqrt{LC}$$

We find that:

$$\frac{\beta}{\omega L} = \frac{\omega\sqrt{LC}}{\omega L} = \sqrt{\frac{C}{L}}$$

Thus, we come to the **startling** conclusion that:

$$\frac{V^+(z)}{I^+(z)} = \sqrt{\frac{L}{C}} \quad \text{and} \quad \frac{-V^-(z)}{I^-(z)} = \sqrt{\frac{L}{C}}$$

Q: *What's so startling about **this** conclusion?*

A: Note that although the magnitude and phase of each propagating wave is a **function** of transmission line **position** z (e.g., $V^+(z)$ and $I^+(z)$), the **ratio** of the voltage and current of **each wave** is independent of position—a **constant** with respect to position z!

Although $V_0^{\pm}$ and $I_0^{\pm}$ are determined by **boundary conditions** (i.e., what's connected to either end of the transmission line), the **ratio** $V_0^{\pm}/I_0^{\pm}$ is determined by the parameters of the transmission line **only** (*R, L, G, C*).

→ This ratio is an important **characteristic of a transmission line**, called its **Characteristic Impedance Z_0**.

$$Z_0 \doteq \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \sqrt{\frac{L}{C}}$$

We can therefore describe the current and voltage along a transmission line as:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}$$

or equivalently:

$$V(z) = Z_0 I_0^+ e^{-j\beta z} - Z_0 I_0^- e^{+j\beta z}$$

$$I(z) = I_0^+ e^{-j\beta z} + I_0^- e^{+j\beta z}$$

Note that instead of characterizing a transmission line with **real** parameters L and C, we can (and typically do!) describe a lossless transmission line using real parameters Z_0 and β.

The Propagation Constant β

Recall that the activity along a transmission line can be expressed in terms of two functions, functions that we have described as **wave** functions:

$$V^{+}(z) = V_0^{+}\, e^{-j\beta z}$$

$$V^{-}(z) = V_0^{-}\, e^{+j\beta z}$$

where β is a real constant with value:

$$\beta = \omega\sqrt{LC}$$

Q: *What **is** this constant β? What does it **physically** represent?*

A: Remember, a complex function can be expressed in terms of its **magnitude** and **phase**:

$$f(z) = |f(z)|\, e^{j\phi_f(z)}$$

Thus:

$$|V^{+}(z)| = |V_0^{+}| \qquad \phi^{+}(z) = -\beta z + \phi_0^{+}$$

$$|V^{-}(z)| = |V_0^{-}| \qquad \phi^{-}(z) = +\beta z + \phi_0^{-}$$

Therefore, $-\beta z + \phi_0^+$ represents the relative **phase** of wave $V^+(z)$; a **function** of transmission line **position** z. Since phase ϕ is expressed in **radians**, and z is distance (in meters), the value β must have **units** of:

$$\beta = \frac{\phi}{z} \qquad \frac{\text{radians}}{\text{meter}}$$

The **wavelength** λ of the propagating wave is defined as the **distance** $\Delta z_{2\pi}$ over which the relative phase changes by 2π **radians**. So:

$$2\pi = \phi(z + \Delta z_{2\pi}) - \phi(z) = \beta \, \Delta z_{2\pi} = \beta \lambda$$

or, rearranging:

$$\boxed{\beta = \frac{2\pi}{\lambda}}$$

Thus, the value β is thus essentially a **spatial frequency**, in the same way that ω is a temporal frequency:

$$\omega = \frac{2\pi}{T}$$

where T is the **time** required for the phase of the oscillating signal to change by a value of 2π radians, i.e.:

$$\omega T = 2\pi$$

Note that this time is the **period** of a sinewave, and related to its **frequency** in Hertz (cycles/second) as:

$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

Q: *So, just how **fast** does this wave propagate down a transmission line?*

We describe wave velocity in terms of its **phase velocity**—in other words, how **fast** does a specific value of absolute phase ϕ seem to **propagate** down the transmission line.

Since velocity is change in distance with respect to **time**, we need to first express our propagating wave in its real form:

$$\begin{aligned} v^+(z,t) &= Re\left\{V^+(z)\,e^{-j\omega t}\right\} \\ &= \left|V_0^+\right| cos\left(\omega t - \beta z + \phi_0^+\right) \end{aligned}$$

Thus, the absolute phase is a function of **both** time and frequency:

$$\phi^+(z,t) = \omega t - \beta z + \phi_0^+$$

Now let's set this phase to some **arbitrary** value of ϕ_c radians.

$$\omega t - \beta z + \phi_0^+ = \phi_c$$

For **every** time t, there is **some** location z on a transmission line that has this phase value ϕ_c. That location is evidently:

$$z = \frac{\omega t + \phi_0^+ - \phi_c}{\beta}$$

Note as **time increases**, so too does the **location** z on the line where $\phi^+(z,t) = \phi_c$.

The **velocity** v_p at which this phase point moves down the line can be determined as:

$$v_p = \frac{dz}{dt} = \frac{d\left(\frac{\omega t + \phi_0^+ - \phi_c}{\beta}\right)}{dt} = \frac{\omega}{\beta}$$

This wave velocity is the **velocity of the propagating wave**!

Note that the value:

$$\frac{v_p}{\lambda} = \frac{\omega}{\beta}\frac{\beta}{2\pi} = \frac{\omega}{2\pi} = f$$

and thus we can conclude that:

$$v_p = f\lambda$$

as well as:

$$\beta = \frac{\omega}{v_p}$$

Q: *But these results were derived for the $V^+(z)$ wave; what about the* ***other*** *wave ($V^-(z)$)?*

A: The results are essentially the **same**, as each wave depends on the same value β.

The only **subtle difference** comes when we evaluate the phase velocity. For the wave $V^{-}(z)$, we find:

$$\phi^{-}(z,t) = \omega t + \beta z + \phi_0^{-}$$

Note the **plus sign** associated with βz!

We thus find that some arbitrary phase value will be located at location:

$$z = \frac{-\phi_0^{-} + \phi_c - \omega t}{\beta}$$

Note now that an **increasing time** will result in a **decreasing** value of **position** z. In other words this wave is propagating in the direction of decreasing position z—in the **opposite** direction of the $V^{+}(z)$ wave!

This is **further** verified by the derivative:

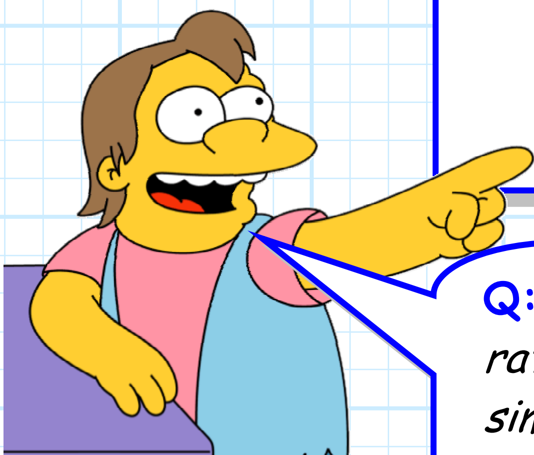
$$v_p = \frac{dz}{dt} = \frac{d\left(\dfrac{-\phi_0^{-} + \phi_c - \omega t}{\beta}\right)}{dt} = -\frac{\omega}{\beta}$$

Where the **minus sign** merely means that the wave propagates in the $-z$ direction. Otherwise, the **wavelength** and **velocity** of the two waves are **precisely** the same!

Line Impedance

Now let's define **line impedance** $Z(z)$, a **complex** function which is simply the ratio of the complex line **voltage** and complex line **current**:

$$Z(z) = \frac{V(z)}{I(z)}$$



Q: *Hey! I know what this is! The ratio of the voltage to current is simply the **characteristic impedance** Z_0, right ???*

A: NO! The line impedance $Z(z)$ is (generally speaking) **NOT** the transmission line **characteristic** impedance Z_0 !!!

→ It is **unfathomably important** that you understand this!!!!

To see why, recall that:

$$V(z) = V^{+}(z) + V^{-}(z)$$

And that:

$$I(z) = \frac{V^+(z) - V^-(z)}{Z_0}$$

Therefore:

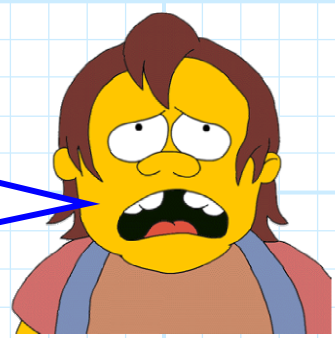
$$Z(z) = \frac{V(z)}{I(z)} = Z_0\left(\frac{V^+(z) + V^-(z)}{V^+(z) - V^-(z)}\right) \neq Z_0$$

Or, more specifically, we can write:

$$Z(z) = Z_0\left(\frac{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}}{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}}\right)$$

Q: *I'm confused! Isn't:*

$$V^+(z)/I^+(z) = Z_0 \text{ ???}$$



A: Yes! That **is** true! The ratio of the voltage to current for **each** of the two propagating waves is $\pm Z_0$. However, the ratio of the **sum** of the two voltages to the **sum** of the two currents is **not** equal to Z_0 (generally speaking)!

This is actually confirmed by the equation above. Say that $V^-(z) = 0$, so that only **one** wave ($V^+(z)$) is propagating on the line.

In this case, the ratio of the **total** voltage to the total current is simply the ratio of the voltage and current of the **one** remaining wave—the **characteristic impedance** Z_0!

$$Z(z) = \frac{V(z)}{I(z)} = Z_0\left(\frac{V^+(z)}{V^+(z)}\right) = \frac{V^+(z)}{I^+(z)} = Z_0 \quad (\text{when } V^-(z) = 0)$$

Q: *So, it appears to me that characteristic impedance Z_0 is a **transmission line parameter**, depending **only** on the transmission line values L and C.*

*Whereas **line impedance** is $Z(z)$ depends the magnitude and phase of the two propagating waves $V^+(z)$ and $V^-(z)$--values that depend **not only** on the transmission line, but also on the two things **attached** to either **end** of the transmission line!*

Right !?

A: Exactly! Moreover, note that characteristic impedance Z_0 is simply a **number**, whereas line impedance $Z(z)$ is a **function** of position (z) on the transmission line.

The Reflection Coefficient

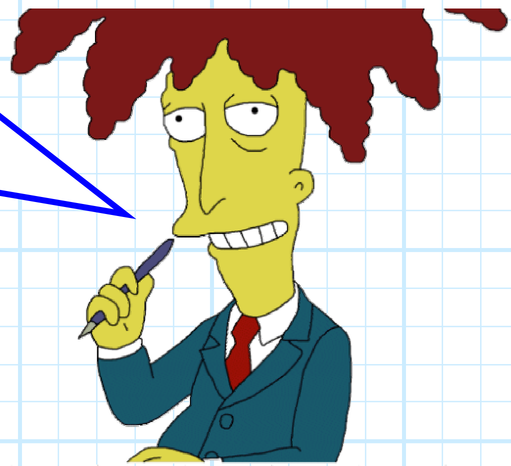
So, we know that the transmission line **voltage** $V(z)$ and the transmission line **current** $I(z)$ can be related by the **line impedance** $Z(z)$:

$$V(z) = Z(z)\,I(z)$$

or equivalently:

$$I(z) = \frac{V(z)}{Z(z)}$$

Q: *Piece of cake! I fully understand the concepts of **voltage**, **current** and **impedance** from my **circuits** classes. Let's move on to something more important (or, at the very least, more **interesting**).*



Expressing the "activity" on a transmission line in terms of **voltage**, **current** and **impedance** is of course **perfectly** valid.

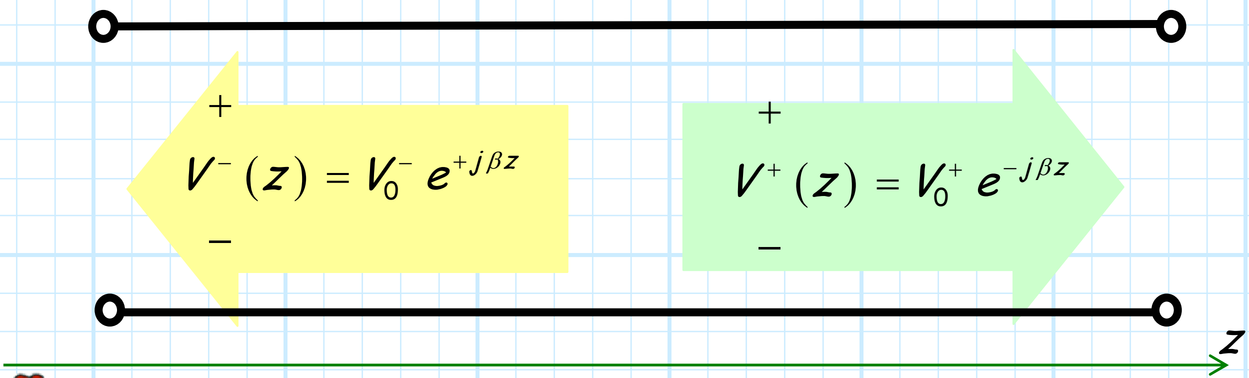
However, let us look **closer** at the expression for each of these quantities:

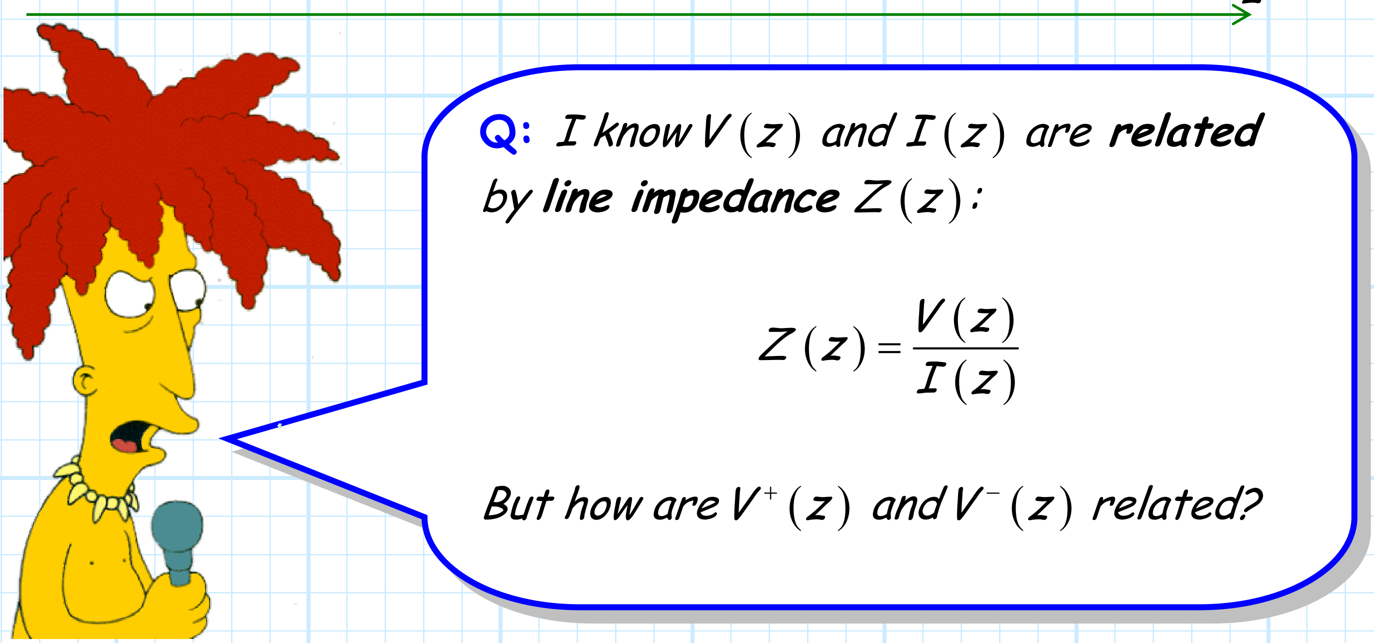
$$V(z) = V^+(z) + V^-(z)$$

$$I(z) = \frac{V^+(z) - V^-(z)}{Z_0}$$

$$Z(z) = Z_0\left(\frac{V^+(z) + V^-(z)}{V^+(z) - V^-(z)}\right)$$

It is evident that we can **alternatively** express all "activity" on the transmission line in terms of the two transmission line **waves** $V^+(z)$ and $V^-(z)$.





A: Similar to line impedance, we can define a new parameter—the **reflection coefficient** $\Gamma(z)$—as the **ratio** of the two quantities:

$$\Gamma(z) \doteq \frac{V^-(z)}{V^+(z)} \quad \Rightarrow \quad V^-(z) = \Gamma(z) V^+(z)$$

More specifically, we can express $\Gamma(z)$ as:

$$\Gamma(z) = \frac{V_0^- \, e^{+j\beta z}}{V_0^+ \, e^{-j\beta z}} = \frac{V_0^-}{V_0^+} \; e^{+j2\beta z}$$

Note then, the value of the reflection coefficient at z=0 is:

$$\Gamma(z=0) = \frac{V^-(z=0)}{V_0^+(z=0)} \; e^{+j2\beta(0)} = \frac{V_0^-}{V_0^+}$$

We define this value as Γ_0, where:

$$\Gamma_0 \doteq \Gamma(z=0) = \frac{V_0^-}{V_0^+}$$

Note then that we can alternatively write $\Gamma(z)$ as:

$$\Gamma(z) = \Gamma_0 \; e^{+j2\beta z}$$

So now we have **two different** but equivalent ways to describe transmission line activity!

We can use (total) **voltage** and **current**, related by **line impedance**:

$$Z(z) = \frac{V(z)}{I(z)} \quad \therefore \quad V(z) = Z(z)\, I(z)$$

Or, we can use the two propagating **voltage waves**, related by the **reflection coefficient**:

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} \quad \therefore \quad V^-(z) = \Gamma(z)\, V^+(z)$$

These are **equivalent** relationships—we can use **either** when describing a transmission line.

*Based on your **circuits** experience, you might well be **tempted** to always use the **first** relationship. However, we will find it useful (as well as simple) indeed to describe activity on a transmission line in terms of the **second** relationship—in terms of the **two** propagating transmission line **waves**!*