B. The Terminated, Lossless Transmission Line

We now know that a **lossless** transmission line is **completely** characterized by **real** constants Z_0 and β.

Likewise, the **2 waves** propagating on a transmission line are **completely** characterized by **complex** constants V_0^+ and V_0^-.

Q: *Z_0 and β are determined from L, C, and ω. How do we find V_0^+ and V_0^- ?*

A: Apply **Boundary Conditions**!

Every transmission line has **2** "boundaries"

1) At one **end** of the transmission line.
2) At the **other** end of the transmission line!

Typically, there is a **source** at one end of the line, and a **load** at the other.

→ The purpose of the transmission line is to get power **from** the source, **to** the load!

Let's apply the **load** boundary condition!

HO: The Terminated, Lossless Transmission Line

HO: Special Values of Load Impedance

Q: *So the line impedance at the* ***end*** *of a line must be load impedance* Z_L *(i.e.,* $Z(z=z_L)=Z_L$*); what is the line impedance at the* ***beginning*** *of the line (i.e.,* $Z(z=z_L-\ell)=?$ *)?*

A: The **input impedance !**

HO: Transmission Line Input Impedance

Q: *You said the purpose of the transmission line is to transfer* ***E.M. energy*** *from the source to the load. Exactly how much* ***power*** *is flowing in the transmission line, and how much is* ***delivered*** *to the load?*

A: *HO: Power Flow and Return Loss*

Note that we can **specify** a load with:

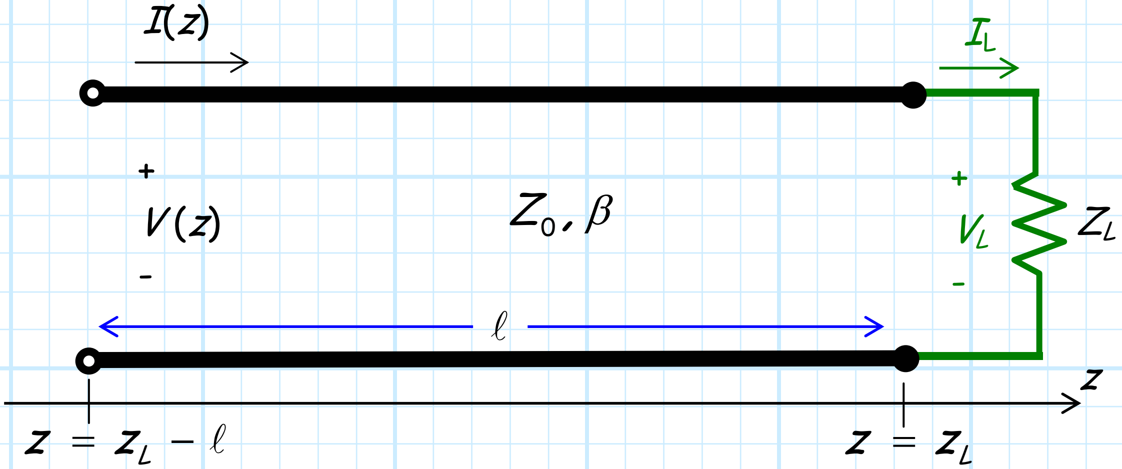
1) its impedance Z_L
2) its reflection coefficient Γ_L
3) return loss

A fourth alternative is **VSWR**.

HO: VSWR

The Terminated, Lossless Transmission Line

Now let's **attach** something to our transmission line. Consider a **lossless** line, length ℓ, terminated with a **load** Z_L.



Q: *What is the **current** and **voltage** at each and **every** point on the transmission line (i.e., what is $I(z)$ and $V(z)$ for **all** points z where $z_L - \ell \le z \le z_L$?)?*

A: To find out, we must apply **boundary conditions**!

In other words, at the **end** of the transmission line ($z = z_L$)—where the load is **attached**—we have **many** requirements that **all** must be satisfied!

1. To begin with, the voltage and current ($I(z=z_L)$ and $V(z=z_L)$) must be consistent with a valid **transmission line solution**:

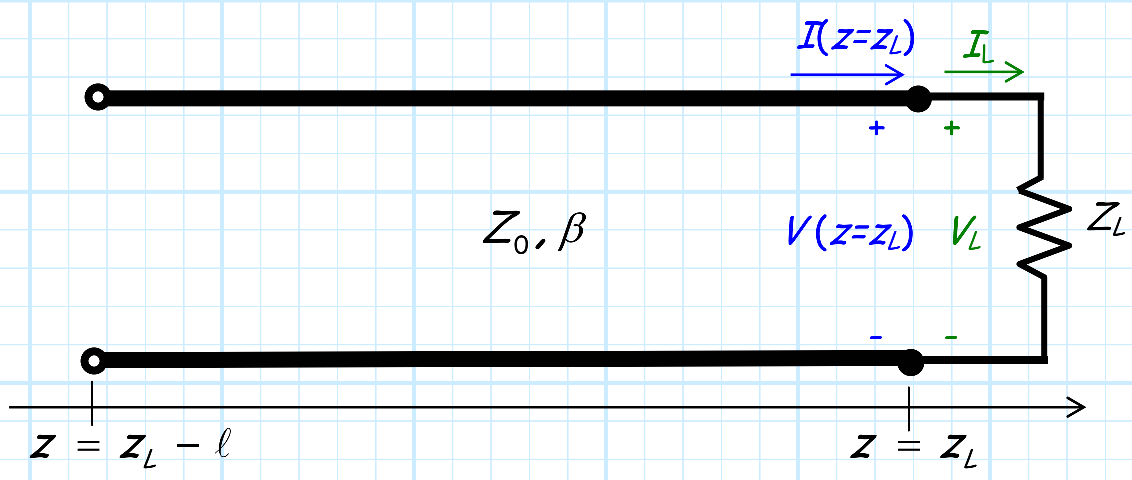
$$
\begin{aligned}
V(z=z_L) &= V^+(z=z_L) + V^-(z=z_L) \\
&= V_0^+ e^{-j\beta z_L} + V_0^- e^{+j\beta z_L}
\end{aligned}
$$

$$
\begin{aligned}
I(z=z_L) &= \frac{V_0^+(z=z_L)}{Z_0} - \frac{V_0^-(z=z_L)}{Z_0} \\
&= \frac{V_0^+}{Z_0} e^{-j\beta z_L} - \frac{V_0^-}{Z_0} e^{+j\beta z_L}
\end{aligned}
$$

2. Likewise, the load voltage and current must be related by **Ohm's law**:

$$V_L = Z_L I_L$$

3. Most importantly, we recognize that the values $I(z=z_L)$, $V(z=z_L)$ and I_L, V_L are **not** independent, but in fact are strictly related by **Kirchoff's Laws**!



From KVL and KCL we find these requirements:

$$V(z = z_L) = V_L$$

$$I(z = z_L) = I_L$$

Combining these equations and boundary conditions, we find that:

$$V_L = Z_L \, I_L$$

$$V(z = z_L) = Z_L \, I(z = z_L)$$

$$V^+(z = z_L) + V^-(z = z_L) = \frac{Z_L}{Z_0}\left(V^+(z = z_L) - V^-(z = z_L)\right)$$

Rearranging, we can conclude:

$$\frac{V^-(z = z_L)}{V^+(z = z_L)} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Q: *Hey wait as second! We earlier defined $V^-(z)/V^+(z)$ as* ***reflection coefficient*** $\Gamma(z)$. *How does this relate to the expression* ***above?***

A: Recall that $\Gamma(z)$ is a **function** of transmission line position z. The value $V^-(z = z_L)/V^+(z = z_L)$ is simply the value of function $\Gamma(z)$ **evaluated** at $z = z_L$ (i.e., evaluated at the **end** of the line):

$$\frac{V^{-}(z=z_L)}{V^{+}(z=z_L)} = \Gamma(z=z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

This value is of **fundamental** importance for the terminated transmission line problem, so we provide it with its **own** special symbol (Γ_L)!

$$\Gamma_L \doteq \Gamma(z=z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Q: *Wait! We **earlier** determined that:*

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}$$

so it would seem that:

$$\Gamma_L = \Gamma(z=z_L) = \frac{Z(z=z_L) - Z_0}{Z(z=z_L) + Z_0}$$

***Which** expression is correct??*

A: They **both** are! It is evident that the two expressions:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad \text{and} \qquad \Gamma_L = \frac{Z(z=z_L) - Z_0}{Z(z=z_L) + Z_0}$$

are **equal** if:

$$Z(z=z_L) = Z_L$$

And since we know that from **Ohm's Law**:

$$Z_L = \frac{V_L}{I_L}$$

and from **Kirchoff's Laws**:

$$\frac{V_L}{I_L} = \frac{V(z = z_L)}{I(z = z_L)}$$

and that **line impedance** is:

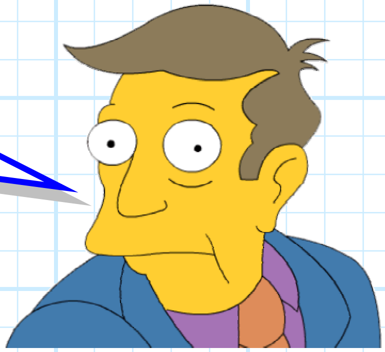
$$\frac{V(z = z_L)}{I(z = z_L)} = Z(z = z_L)$$

we find it apparent that the **line impedance** at the **end** of the transmission line is **equal** to the **load impedance**:

$$Z(z = z_L) = Z_L$$

The above expression is essentially **another** expression of the **boundary condition** applied at the **end** of the transmission line.

Q: *I'm confused! Just what are were we trying to accomplish in this handout?*



A: We are trying to find $V(z)$ and $I(z)$ when a lossless transmission line is terminated by a load Z_L!

We can now determine the value of V_0^- in terms of V_0^+. Since:

$$\Gamma_L = \frac{V^-(z=z_L)}{V^+(z=z_L)} = \frac{V_0^- e^{+j\beta z_L}}{V_0^+ e^{-j\beta z_L}}$$

We find:

$$V_0^- = e^{-2j\beta z_L}\,\Gamma_L\,V_0^+$$

And therefore we find:

$$V^-(z) = \left(e^{-2j\beta z_L}\,\Gamma_L\,V_0^+\right)e^{+j\beta z}$$

$$V(z) = V_0^+\left[e^{-j\beta z} + \left(e^{-2j\beta z_L}\,\Gamma_L\right)e^{+j\beta z}\right]$$

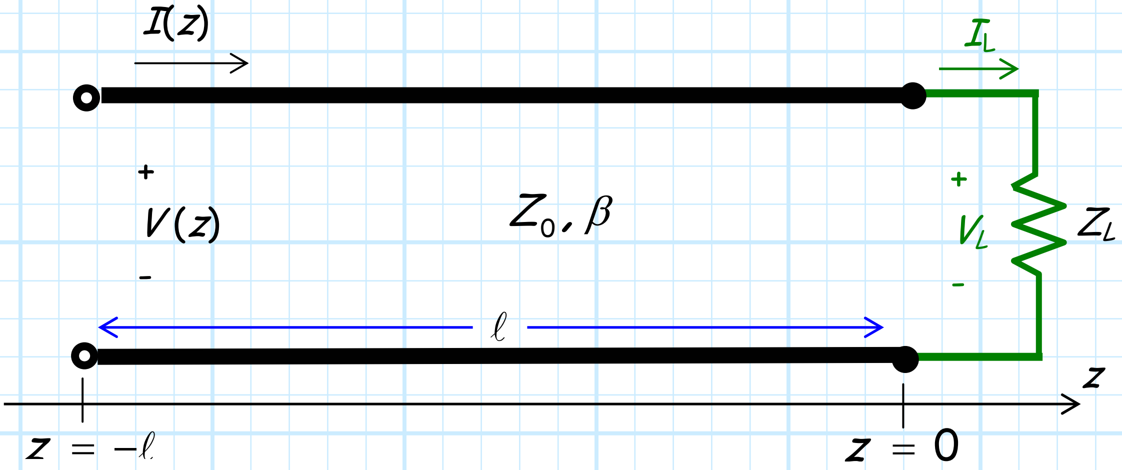
$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} - \left(e^{-2j\beta z_L}\,\Gamma_L\right)e^{+j\beta z}\right]$$

where:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$z_L = 0$

Now, we can further **simplify** our analysis by **arbitrarily** assigning the end point z_L a **zero** value (i.e., $z_L = 0$):



If the load is located at z=0 (i.e., **if** $z_L = 0$), we find that:

$$\begin{aligned} V(z=0) &= V^+(z=0) + V^-(z=0) \\ &= V_0^+ e^{-j\beta(0)} + V_0^- e^{+j\beta(0)} \\ &= V_0^+ + V_0^- \end{aligned}$$

$$\begin{aligned} I(z=0) &= \frac{V_0^+(z=0)}{Z_0} - \frac{V_0^-(z=0)}{Z_0} \\ &= \frac{V_0^+}{Z_0} e^{-j\beta(0)} - \frac{V_0^-}{Z_0} e^{+j\beta(0)} \\ &= \frac{V_0^+ - V_0^-}{Z_0} \end{aligned}$$

$$Z(z=0) = Z_0 \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \right)$$

Likewise, it is apparent that **if** $z_L = 0$, Γ_L and Γ_0 are the same:

$$\Gamma_L = \Gamma(z = z_L) = \frac{V^-(z=0)}{V^+(z=0)} = \frac{V_0^-}{V_0^+} = \Gamma_0$$

Therefore:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_0$$

Thus, we can write the line current and voltage simply as:

$$V(z) = V_0^+\left[e^{-j\beta z} + \Gamma_0\, e^{+j\beta z}\right]$$

$$\left[\text{for } z_L = 0\right]$$

$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} - \Gamma_0\, e^{+j\beta z}\right]$$

Q: *But, how do we determine* V_0^+ *??*

A: We require a **second** boundary condition to determine V_0^+. The only boundary left is at the **other end** of the transmission line. Typically, a **source** of some sort is located there. This makes physical sense, as something must generate the **incident** wave !

Special Values of Load Impedance

It's interesting to note that the load Z_L enforces a boundary condition that explicitly determines neither $V(z)$ nor $I(z)$—but **completely** specifies **line impedance** $Z(z)$!

$$Z(z) = Z_0 \frac{e^{-j\beta z} + \Gamma_L\, e^{+j\beta z}}{e^{-j\beta z} - \Gamma_L\, e^{+j\beta z}} = Z_0 \frac{Z_L \cos\beta z - jZ_0 \sin\beta z}{Z_0 \cos\beta z - jZ_L \sin\beta z}$$

$$\Gamma(z) = \Gamma_L\, e^{+j2\beta z} = \frac{Z_L - Z_0}{Z_L + Z_0}\, e^{+j2\beta z}$$

Likewise, the load boundary condition leaves $V^{+}(z)$ and $V^{-}(z)$ undetermined, but **completely** determines **reflection coefficient function** $\Gamma(z)$!

Let's look at some **specific** values of load impedance $Z_L = R_L + jX_L$ and see what functions $Z(z)$ and $\Gamma(z)$ result!

1. $Z_L = Z_0$

In this case, the **load** impedance is **numerically** equal to the **characteristic** impedance of the transmission line. Assuming the line is **lossless**, then Z_0 is **real**, and thus:

$$R_L = Z_0 \qquad \text{and} \qquad X_L = 0$$

It is evident that the resulting load reflection coefficient is **zero**:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

This result is very interesting, as it means that there is **no reflected wave** $V^-(z)$!

Thus, the **total** voltage and current along the transmission line is simply voltage and current of the **incident** wave:

$$V(z) = V^+(z) = V_0^+ e^{-j\beta z}$$

$$I(z) = I^+(z) = \frac{V_0^+}{Z_0} e^{-j\beta z}$$

Meaning that the **line** impedance is likewise **numerically** equal to the **characteristic** impedance of the transmission line for **all** line position z:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{V_0^+ e^{-j\beta z}}{V_0^+ e^{-j\beta z}} = Z_0$$

And likewise, the reflection coefficient is **zero** at **all** points along the line:

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{0}{V^+(z)} = 0$$

We call this condition (when $Z_L = Z_0$) the **matched** condition, and the load $Z_L = Z_0$ a **matched load**.

2. $Z_L = jX_L$

For this case, the load impedance is **purely reactive** (e.g. a capacitor of inductor), the real (resistive) portion of the load is zero:

$$R_L = 0$$

The resulting load reflection coefficient is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{jX_L - Z_0}{jX_L + Z_0}$$

Given that $Z_0$is real (i.e., the line is **lossless**), we find that this load reflection coefficient is generally some **complex** number.

We **can** rewrite this value explicitly in terms of its **real** and **imaginary** part as:

$$\Gamma_L = \frac{jX_L - Z_0}{jX_L + Z_0} = \left(\frac{X_L^2 - Z_0^2}{X_L^2 + Z_0^2}\right) + j\left(\frac{2\,Z_0\,X_L}{X_L^2 + Z_0^2}\right)$$

*Yuck! This **isn't** much help!*

et's **instead** write this complex value Γ_L in terms of its **magnitude** and **phase**. For **magnitude** we find a much more straightforward result!

$$|\Gamma_L|^2 = \frac{|jX_L - Z_0|^2}{|jX_L + Z_0|^2} = \frac{X_L^2 + Z_0^2}{X_L^2 + Z_0^2} = 1$$

Its magnitude is **one**! Thus, we find that for reactive loads, the reflection coefficient can be simply expressed as:

$$\Gamma_L = e^{j\theta_\Gamma}$$

where

$$\theta_\Gamma = tan^{-1}\left[\frac{2\,Z_0\,X_L}{X_L^2 - Z_0^2}\right]$$

We can therefore conclude that for a **reactive load**:

$$V_0^- = e^{j\theta_\Gamma}\, V_0^+$$

As a result, the **total** voltage and current along the transmission line is simply (assuming $z_L = 0$):

$$\begin{aligned} V(z) &= V_0^+\left(e^{-j\beta z} + e^{+j\theta_L} e^{+j\beta z}\right) \\ &= V_0^+\ e^{+j\theta_\Gamma/2}\left(e^{-j(\beta z + \theta_\Gamma/2)} + e^{+j(\beta z + \theta_\Gamma/2)}\right) \\ &= 2V_0^+\ e^{+j\theta_\Gamma/2}\ \cos\left(\beta z + \theta_\Gamma/2\right) \end{aligned}$$

$$
\begin{aligned}
I(z) &= \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - e^{+j\beta z}\right) \\
&= \frac{V_0^+}{Z_0} e^{+j\theta_L/2}\left(e^{-j(\beta z+\theta_L/2)} - e^{+j(\beta z+\theta_L/2)}\right) \\
&= -j\,\frac{2V_0^+}{Z_0} e^{+j\theta_L/2} \sin\left(\beta z + \theta_L/2\right)
\end{aligned}
$$

Meaning that the **line impedance** can be written in terms of a trigonometric function:

$$
Z(z) = \frac{V(z)}{I(z)} = j\,Z_0 \cot\left(\beta z + \theta_\Gamma/2\right)
$$

Note that this impedance is **purely reactive**— $V(z)$ and $I(z)$ are 90° out of phase!

Finally, the reflection coefficient **function** is:

$$
\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \frac{V_0^+ e^{+j\theta_\Gamma} e^{+j\beta z}}{V_0^+ e^{-j\beta z}} = e^{+j2(\beta z+\theta_\Gamma/2)}
$$

Meaning that for purely reactive loads:

$$
\left|\Gamma(z)\right| = \left|e^{+j2(\beta z+\theta_\Gamma/2)}\right| = 1
$$

In other words, the **magnitude** reflection coefficient function is equal to one—at each and **every** point on the transmission line.

3. $Z_L = R_L$

For this case, the load impedance is **purely real** (e.g. a **resistor**), and thus there is no reactive component:

$$X_L = 0$$

The resulting **load reflection coefficient** is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R - Z_0}{R + Z_0}$$

Given that Z_0 is real (i.e., the line is **lossless**), we find that this load reflection coefficient must be a purely **real** value! In other words:

$$Re\{\Gamma_L\} = \frac{R - Z_0}{R + Z_0} \qquad \text{Im}\{\Gamma_L\} = 0$$

So a real-valued load Z_L results in a real valued load reflection coefficient G_L .

Now let's consider the line impedance $Z(z)$ and reflection coefficient function $\Gamma(z)$.

Q: *I bet I know the answer to this one! We know that a purely imaginary (i.e., reactive) load results in a purely reactive line impedance.*

Thus, a purely real (i.e., resistive) load will result in a purely resistive line impedance, right??

A: NOPE! The line impedance resulting from a real load is complex—it has both real and imaginary components!

Thus the line impedance, as well as reflection coefficient function, **cannot** be further simplified for the case where $Z_L = R_L$.

Q: *Why is that?*

A: Remember, a **lossless** transmission line has series inductance and shunt capacitance **only**. In other words, a length of lossless transmission line is a **purely reactive** device (it absorbs **no** energy!).

* If we attach a **purely reactive** load at the end of the transmission line, we still have a **completely** reactive system (load and transmission line). Because this system has **no** resistive (i.e., real) component, the general expressions for line impedance, line voltage, etc. can be significantly **simplified**.

* However, if we attach a **purely real** load to our reactive transmission line, we now have a **complex** system, with **both** real and imaginary (i.e., resistive and reactive) components. This **complex** case is exactly what our general expressions **already** describes—**no** further simplification is possible!

4. $Z_L = R_L + jX_L$

Now, let's look at the **general** case, where the **load** has both a **real** (resitive) and **imaginary** (reactive) component.

Q: *Haven't we **already** determined all the **general** expressions (e.g., $\Gamma_L, V(z), I(z), Z(z), \Gamma(z)$) for this general case? Is there **anything** else left to be determined?*

A: There is **one** last thing we need to discuss. It seems trivial, but its ramifications are **very** important!

For you see, the "general" case is **not**, in reality, quite so general. Although the reactive component of the load can be **either** positive or negative ($-\infty < X_L < \infty$), the resistive component of a passive load **must** be positive ($R_L > 0$)—there's **no** such thing as **negative** resistor!

This leads to one **very** important and useful result. Consider the **load reflection coefficient**:

$$\begin{aligned}\Gamma_L &= \frac{Z_L - Z_0}{Z_L + Z_0} \\ &= \frac{(R_L + jX_L) - Z_0}{(R_L + jX_L) + Z_0} \\ &= \frac{(R_L - Z_0) + jX_L}{(R_L + Z_0) + jX_L}\end{aligned}$$

Now let's look at the **magnitude** of this value:

$$
\begin{aligned}
|\Gamma_L|^2 &= \left|\frac{(R_L - Z_0) + jX_L}{(R_L + Z_0) + jX_L}\right|^2 \\
&= \frac{(R_L - Z_0)^2 + X_L^2}{(R_L + Z_0)^2 + X_L^2} \\
&= \frac{\left(R_L^2 - 2R_L Z_0 + Z_0^2\right) + X_L^2}{\left(R_L^2 + 2R_L Z_0 + Z_0^2\right) + X_L^2} \\
&= \frac{\left(R_L^2 + Z_0^2 + X_L^2\right) - 2R_L Z_0}{\left(R_L^2 + Z_0^2 + X_L^2\right) + 2R_L Z_0}
\end{aligned}
$$

It is apparent that since both R_L and Z_0 are **positive**, the **numerator** of the above expression must be **less** than (or equal to) the **denominator** of the above expression.

→ In other words, the magnitude of the load reflection coefficient is always **less** than or equal to one!

$$
|\Gamma_L| \leq 1 \quad (\text{for } R_L \geq 0)
$$

Moreover, we find that this means the reflection coefficient **function** likewise always has a magnitude **less** than or equal to one, for **all** values of position *z*.

$$|\Gamma(z)| \leq 1 \quad \text{(for all } z\text{)}$$

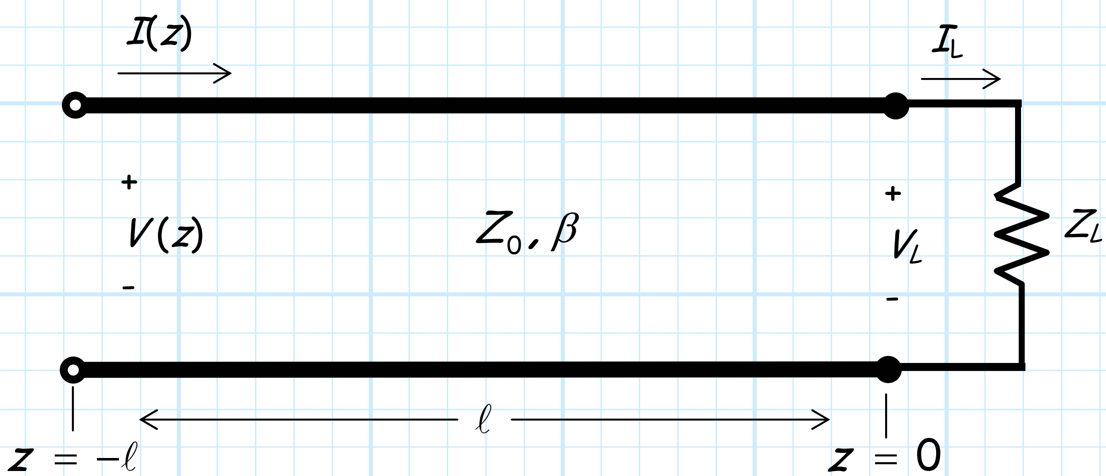
Which means, of course, that the **reflected** wave will always have a magnitude **less** than that of the **incident** wave magnitude:

$$|V^-(z)| \leq |V^+(z)| \qquad \text{(for all } z\text{)}$$

We will find out later that this result is consistent with **conservation of energy**—the reflected wave from a passive load **cannot** be larger than the wave incident on it.

Transmission Line Input Impedance

Consider a **lossless** line, length ℓ, terminated with a **load** Z_L.



Let's determine the **input impedance** of this line!

Q: *Just what do you mean by **input** impedance?*

A: The input impedance is simply the line impedance seen at the **beginning** ($z = -\ell$) of the transmission line, i.e.:

$$Z_{in} = Z(z = -\ell) = \frac{V(z = -\ell)}{I(z = -\ell)}$$

Note Z_{in} equal to **neither** the load impedance Z_L **nor** the characteristic impedance Z_0!

$$Z_{in} \neq Z_L \quad \text{and} \quad Z_{in} \neq Z_0$$

To determine exactly what Z_{in} is, we first must determine the voltage and current at the **beginning** of the transmission line ($z = -\ell$).

$$V(z=-\ell) = V_0^+\left[e^{+j\beta\ell} + \Gamma_L e^{-j\beta\ell}\right]$$

$$I(z=-\ell) = \frac{V_0^+}{Z_0}\left[e^{+j\beta\ell} - \Gamma_L e^{-j\beta\ell}\right]$$

Therefore:

$$Z_{in} = \frac{V(z=-\ell)}{I(z=-\ell)} = Z_0\left(\frac{e^{+j\beta\ell} + \Gamma_L e^{-j\beta\ell}}{e^{+j\beta\ell} - \Gamma_L e^{-j\beta\ell}}\right)$$

We can explicitly write Z_{in} in terms of load Z_L using the previously determined relationship:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Combining these two expressions, we get:

$$\begin{aligned} Z_{in} &= Z_0 \frac{(Z_L + Z_0)e^{+j\beta\ell} + (Z_L - Z_0)e^{-j\beta\ell}}{(Z_L + Z_0)e^{+j\beta\ell} - (Z_L - Z_0)e^{-j\beta\ell}} \\ &= Z_0\left(\frac{Z_L\left(e^{+j\beta\ell} + e^{-j\beta\ell}\right) + Z_0\left(e^{+j\beta\ell} - e^{-j\beta\ell}\right)}{Z_L\left(e^{+j\beta\ell} + e^{-j\beta\ell}\right) - Z_0\left(e^{+j\beta\ell} - e^{-j\beta\ell}\right)}\right) \end{aligned}$$

Now, recall **Euler's equations**:

$$e^{+j\beta\ell} = \cos\beta\ell + j\sin\beta\ell$$
$$e^{-j\beta\ell} = \cos\beta\ell - j\sin\beta\ell$$

Using Euler's relationships, we can likewise write the input impedance without the **complex** exponentials:

$$Z_{in} = Z_0 \left(\frac{Z_L \cos\beta\ell + j\, Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\, Z_L \sin\beta\ell} \right)$$

$$= Z_0 \left(\frac{Z_L + j\, Z_0 \tan\beta\ell}{Z_0 + j\, Z_L \tan\beta\ell} \right)$$

Note that depending on the values of β, Z_0 and ℓ, the input impedance can be **radically** different from the load impedance Z_L!

Special Cases

Now let's look at the Z_{in} for some important **load** impedances and **line lengths**.

→ You should commit these results to **memory**!

1. $\ell = \lambda/2$

If the length of the transmission line is exactly **one-half** wavelength ($\ell = \lambda/2$), we find that:

$$\beta\ell = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi$$

meaning that:

$$\cos\beta\ell = \cos\pi = -1 \qquad \text{and} \qquad \sin\beta\ell = \sin\pi = 0$$

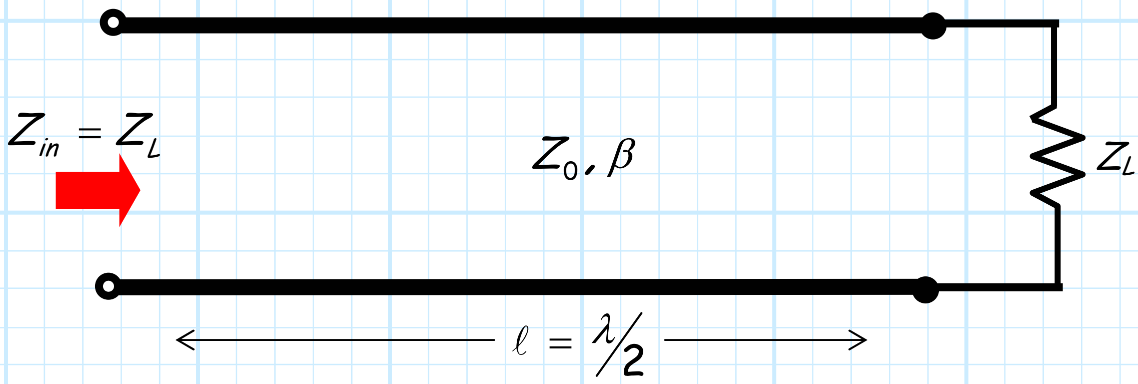
and therefore:

$$Z_{in} = Z_0 \left(\frac{Z_L \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j\, Z_L \sin \beta\ell} \right)$$

$$= Z_0 \left(\frac{Z_L\,(-1) + j\, Z_L\,(0)}{Z_0\,(-1) + j\, Z_L\,(0)} \right)$$

$$= Z_L$$

In other words, if the transmission line is precisely **one-half wavelength** long, the **input** impedance is equal to the **load** impedance, **regardless** of Z_0 or β.



2. $\ell = \lambda/4$

If the length of the transmission line is exactly **one-quarter** wavelength ($\ell = \lambda/4$), we find that:

$$\beta\ell = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$$

meaning that:

$$\cos \beta\ell = \cos\ \pi/2 = 0 \qquad \text{and} \qquad \sin \beta\ell = \sin\ \pi/2 = 1$$

and therefore:

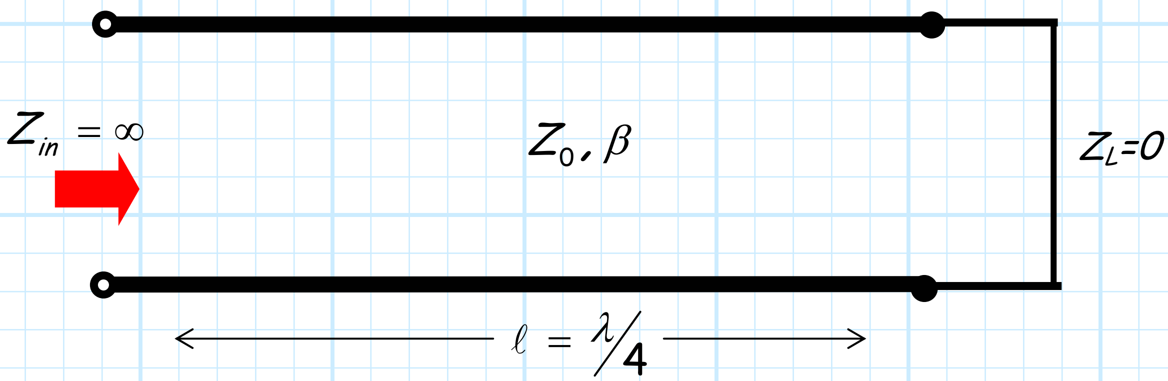
$$Z_{in} = Z_0 \left(\frac{Z_L \cos \beta \ell + j\, Z_0 \sin \beta \ell}{Z_0 \cos \beta \ell + j\, Z_L \sin \beta \ell} \right)$$
$$= Z_0 \left(\frac{Z_L\,(0) + j\, Z_0\,(1)}{Z_0\,(0) + j\, Z_L\,(1)} \right)$$
$$= \frac{(Z_0)^2}{Z_L}$$

In other words, if the transmission line is precisely **one-quarter wavelength** long, the **input** impedance is **inversely** proportional to the **load** impedance.

Think about what this means! Say the load impedance is a **short** circuit, such that $Z_L = 0$. The **input impedance** at beginning of the $\lambda/4$ transmission line is therefore:

$$Z_{in} = \frac{(Z_0)^2}{Z_L} = \frac{(Z_0)^2}{0} = \infty$$

$Z_{in} = \infty$! This is an **open** circuit! The quarter-wave transmission line **transforms** a short-circuit into an open-circuit—and vice versa!

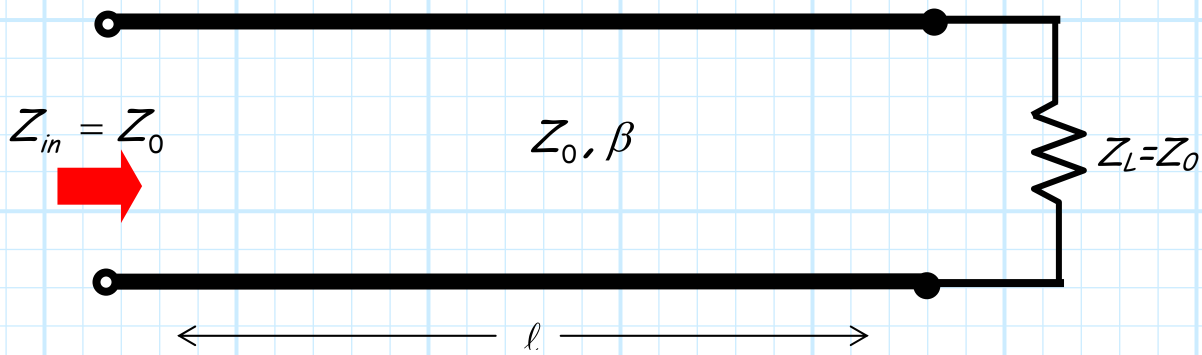


3. $Z_L = Z_0$

If the load is **numerically equal** to the characteristic impedance of the transmission line (a real value), we find that the input impedance becomes:

$$
\begin{aligned}
Z_{in} &= Z_0 \left(\frac{Z_L \cos\beta\ell + j\, Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\, Z_L \sin\beta\ell} \right) \\
&= Z_0 \left(\frac{Z_0 \cos\beta\ell + j\, Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\, Z_0 \sin\beta\ell} \right) \\
&= Z_0
\end{aligned}
$$

In other words, if the **load** impedance is equal to the transmission line **characteristic** impedance, the **input** impedance will be likewise be equal to Z_0 regardless of the transmission line length ℓ.



4. $Z_L = j\, X_L$

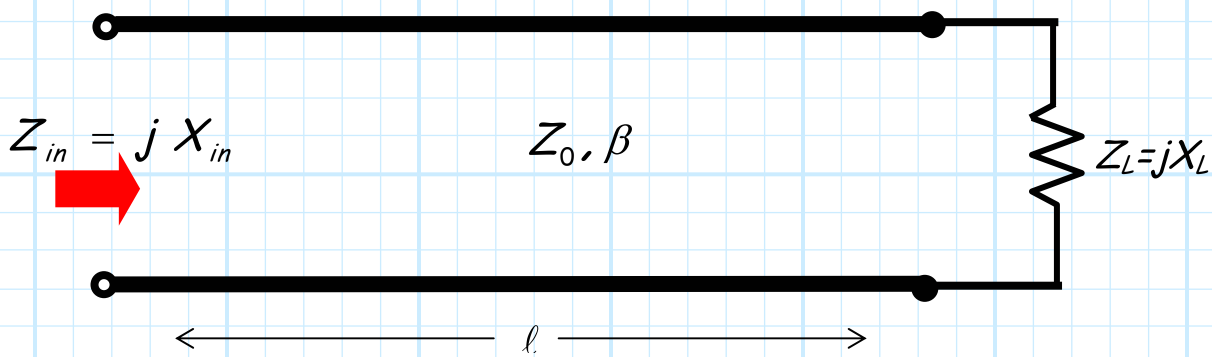
If the load is **purely reactive** (i.e., the resistive component is zero), the input impedance is:

$$Z_{in} = Z_0 \left(\frac{Z_L \cos\beta\ell + j\,Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\,Z_L \sin\beta\ell} \right)$$

$$= Z_0 \left(\frac{j\,X_L \cos\beta\ell + j\,Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j^2\,X_L \sin\beta\ell} \right)$$

$$= j\,Z_0 \left(\frac{X_L \cos\beta\ell + Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell - X_L \sin\beta\ell} \right)$$

In other words, if the load is purely reactive, then the input impedance will **likewise** be purely reactive, **regardless** of the line length ℓ.



Note that the **opposite** is **not** true: even if the load is **purely resistive** ($Z_L = R$), the input impedance will be **complex** (both resistive and reactive components).

Q: *Why is this?*

A:

5. $\ell \ll \lambda$

If the transmission line is **electrically small**—its length ℓ is small with respect to signal wavelength λ--we find that:

$$\beta\ell = \frac{2\pi}{\lambda}\ell = 2\pi\frac{\ell}{\lambda} \approx 0$$

and thus:

$$\cos\beta\ell = \cos 0 = 1 \qquad \text{and} \qquad \sin\beta\ell = \sin 0 = 0$$

so that the input impedance is:

$$\begin{aligned} Z_{in} &= Z_0\left(\frac{Z_L\cos\beta\ell + j\,Z_0\sin\beta\ell}{Z_0\cos\beta\ell + j\,Z_L\sin\beta\ell}\right) \\ &= Z_0\left(\frac{Z_L\,(1) + j\,Z_L\,(0)}{Z_0\,(1) + j\,Z_L\,(0)}\right) \\ &= Z_L \end{aligned}$$

In other words, if the transmission line length is much smaller than a wavelength, the **input** impedance Z_{in} will **always** be equal to the **load** impedance Z_L.

This is the assumption we used in all previous circuits courses (e.g., EECS 211, 212, 312, 412)! In those courses, we assumed that the signal frequency ω is relatively **low**, such that the signal wavelength λ is **very large** ($\lambda \gg \ell$).

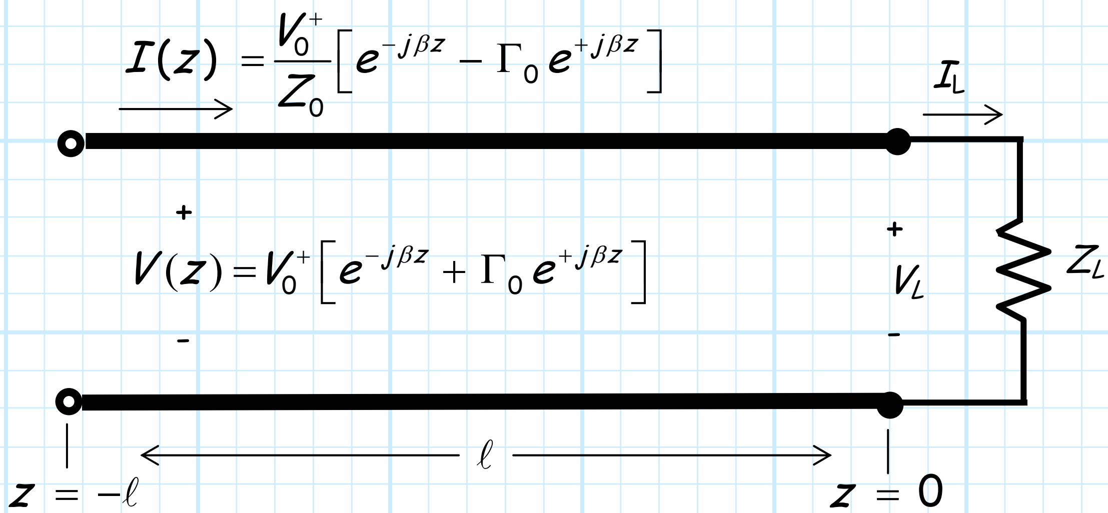
Note also for this case (the electrically short transmission line), the voltage and current at each end of the transmission line are approximately the **same**!

$$V(z=-\ell) \approx V(z=0) \quad \text{and} \quad I(z=-\ell) \approx I(z=0) \quad \text{if} \quad \ell \ll \lambda$$

If $\ell \ll \lambda$, our "wire" behaves **exactly** as it did in EECS 211 !

Power Flow and Return Loss

We have discovered that **two waves propagate** along a transmission line, one in each direction ($V^+(z)$ and $V^-(z)$).



The result is that electromagnetic energy flows along the transmission line at a given rate (i.e., **power**).

Q: *How much power flows along a transmission line, and where does that power go?*

A: We can answer that question by determining the power **absorbed** by the **load**!

The **time average** power absorbed by an impedance Z_L is:

$$\begin{aligned} P_{abs} &= \frac{1}{2}\mathrm{Re}\left\{V_L I_L^*\right\} \\ &= \frac{1}{2}\mathrm{Re}\left\{V(z=0)I(z=0)^*\right\} \\ &= \frac{1}{2Z_0}\mathrm{Re}\left\{\left(V_0^+\left[e^{-j\beta 0}+\Gamma_0 e^{+j\beta 0}\right]\right)\left(V_0^+\left[e^{-j\beta 0}-\Gamma_0 e^{+j\beta 0}\right]\right)^*\right\} \\ &= \frac{\left|V_0^+\right|^2}{2Z_0}\mathrm{Re}\left\{1-\left(\Gamma_0^*-\Gamma_0\right)-\left|\Gamma_0\right|^2\right\} \\ &= \frac{\left|V_0^+\right|^2}{2Z_0}\left(1-\left|\Gamma_0\right|^2\right) \end{aligned}$$

The significance of this result can be seen by **rewriting** the expression as:

$$\begin{aligned} P_{abs} &= \frac{\left|V_0^+\right|^2}{2Z_0}\left(1-\left|\Gamma_0\right|^2\right) \\ &= \frac{\left|V_0^+\right|^2}{2Z_0}-\frac{\left|V_0^+\Gamma_0\right|^2}{2Z_0} \\ &= \frac{\left|V_0^+\right|^2}{2Z_0}-\frac{\left|V_0^-\right|^2}{2Z_0} \end{aligned}$$

The two terms in above expression have a very definite **physical meaning**. The first term is the time-averaged **power of the wave** propagating along the transmission line **toward the load**.

We say that this wave is **incident** on the load:

$$P_{inc} = P_{+} = \frac{\left|V_0^+\right|^2}{2Z_0}$$

Likewise, the second term of the P_{abs} equation describes the **power of the wave** moving in the other direction (**away from the load**). We refer to this as the wave **reflected** from the load:

$$P_{ref} = P_{-} = \frac{\left|V_0^-\right|^2}{2Z_0} = \frac{\left|\Gamma_0\right|^2 \left|V_0^+\right|^2}{2Z_0} = \left|\Gamma_L\right|^2 P_{inc}$$

Thus, the power **absorbed** by the load (i.e., the power **delivered to** the load) is simply:

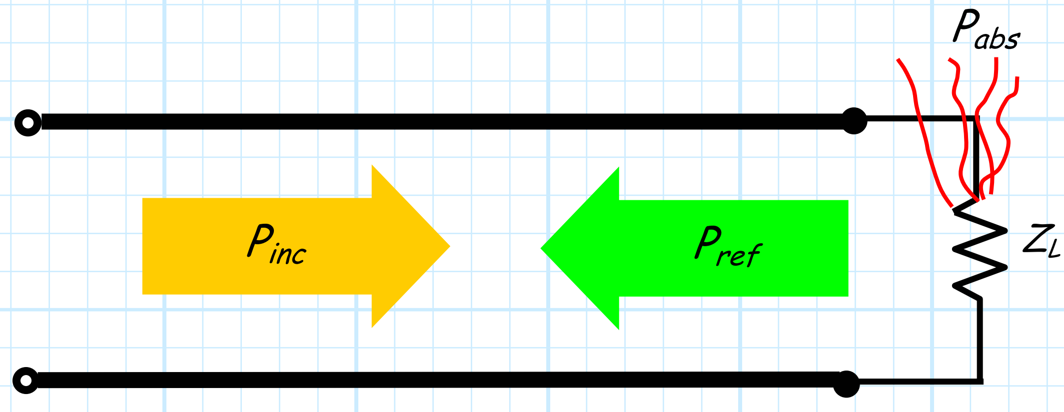
$$P_{abs} = P_{inc} - P_{ref}$$

or, rearranging, we find:

$$P_{inc} = P_{abs} + P_{ref}$$

This equation is simply an expression of the **conservation of energy** !

It says that power flowing **toward** the load (P_{inc}) is either **absorbed** by the load (P_{abs}) or **reflected** back from the load (P_{ref}).



Note that if $\left|\Gamma_L\right|^2 = 1$, then $P_{inc} = P_{ref}$, and therefore **no power** is absorbed by the **load**.

This of course **makes sense** !

The magnitude of the reflection coefficient ($\left|\Gamma_L\right|$) is equal to one **only** when the load impedance is **purely reactive** (i.e., purely imaginary).

Of course, a purely reactive element (e.g., capacitor or inductor) **cannot** absorb any power—**all** the power **must** be reflected!

Return Loss

The **ratio** of the reflected power to the incident power is known as **return loss**. Typically, return loss is expressed in **dB**:

$$R.L. = -10 \log_{10}\left[\frac{P_{ref}}{P_{inc}}\right] = -10 \log_{10}\left|\Gamma_L\right|^2$$

For **example**, if the return loss is **10dB**, then **10%** of the incident power is **reflected** at the load, with the remaining **90%** being **absorbed** by the load—we "lose" 10% of the incident power

Likewise, if the return loss is **30dB**, then **0.1 %** of the incident power is **reflected** at the load, with the remaining **99.9%** being **absorbed** by the load—we "lose" 0.1% of the incident power.

Thus, a **larger** numeric value for return loss **actually** indicates **less** lost power! An **ideal** return loss would be ∞ dB, whereas a return loss of 0 dB indicates that $\left|\Gamma_L\right| = 1$--the load is **reactive**!

VSWR

Consider again the **voltage** along a terminated transmission line, as a function of **position** z:

$$V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma_L e^{+j\beta z} \right]$$

Recall this is a **complex** function, the magnitude of which expresses the **magnitude** of the **sinusoidal signal** at position z, while the phase of the complex value represents the relative **phase** of the sinusoidal signal.

Let's look at the **magnitude** only:

$$\begin{aligned} |V(z)| &= |V_0^+| \, |e^{-j\beta z} + \Gamma_L e^{+j\beta z}| \\ &= |V_0^+| |e^{-j\beta z}| |1 + \Gamma_L e^{+j2\beta z}| \\ &= |V_0^+| |1 + \Gamma_L e^{+j2\beta z}| \end{aligned}$$

ICBST the **largest** value of $|V(z)|$ occurs at the location z where:

$$\Gamma_L e^{+j2\beta z} = |\Gamma_L| + j0$$

while the **smallest** value of $|V(z)|$ occurs at the location z where:

$$\Gamma_L e^{+j2\beta z} = -|\Gamma_L| + j0$$

As a result we can conclude that:

$$|V(z)|_{max} = |V_0^+|\left(1 + |\Gamma_L|\right)$$

$$|V(z)|_{min} = |V_0^+|\left(1 - |\Gamma_L|\right)$$

The ratio of $|V(z)|_{max}$ to $|V(z)|_{min}$ is known as the **Voltage Standing Wave Ratio** (VSWR):

$$\text{VSWR} \doteq \frac{|V(z)|_{max}}{|V(z)|_{min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \qquad \therefore \qquad 1 \leq VSWR \leq \infty$$

Note if $|\Gamma_L| = 0$ (i.e., $Z_L = Z_0$), then VSWR = 1. We find for this case:

$$|V(z)|_{max} = |V(z)|_{min} = |V_0^+|$$

In other words, the voltage magnitude is a **constant** with respect to position *z*.

Conversely, if $|\Gamma_L| = 1$ (i.e., $Z_L = jX$), then VSWR = ∞. We find for **this** case:

$$|V(z)|_{min} = 0 \qquad \text{and} \qquad |V(z)|_{max} = 2|V_0^+|$$

In other words, the voltage magnitude varies **greatly** with respect to position *z*.

As with **return loss**, VSWR is dependent on the **magnitude** of Γ_L (i.e, $|\Gamma_L|$) **only** !