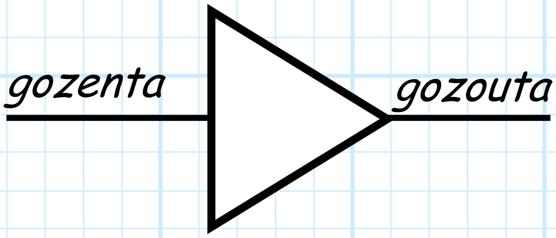
E. Microwave Network Theory

Note that a passive load is a one-port device—a device that can be characterized (at one frequency) by impedance Z_L or load reflection coefficient Γ_L .

However, many microwave devices have **multiple** ports!

Most common are **two-port devices** (e.g., amplifiers and filters), devices with both a gozenta and a gozouta.



Note that a transmission line is also two-port device!

Q: *Are there any known ways to characterize a **multi-port** device?*

A: Yes! **Two** methods are:

1. The **impedance** matrix—a multi-port equivalent of Z_L

2. The **scattering** matrix—a multi-port equivalent of Γ_L

HO: The Impedance Matrix

Q: *You say that the impedance matrix characterizes a **multi-port** device. But is this characterization **helpful**? Can we actually use it to **solve** real problems?*

A: *Example: Using the Impedance Matrix*

Q: *The **impedance** matrix relates the quantities $I(z)$ and $V(z)$, is there an **equivalent** matrix that relates $V^+(z)$ and $V^-(z)$?*

A: Yes! The **scattering matrix** relates the t.l. **waves** entering and exiting a multi-port device!

HO: The Scattering Matrix

Q: *Can the scattering matrix likewise be used to solve real problems?*

A: Of course!

Example: The Scattering Matrix

Example: Scattering Parameters

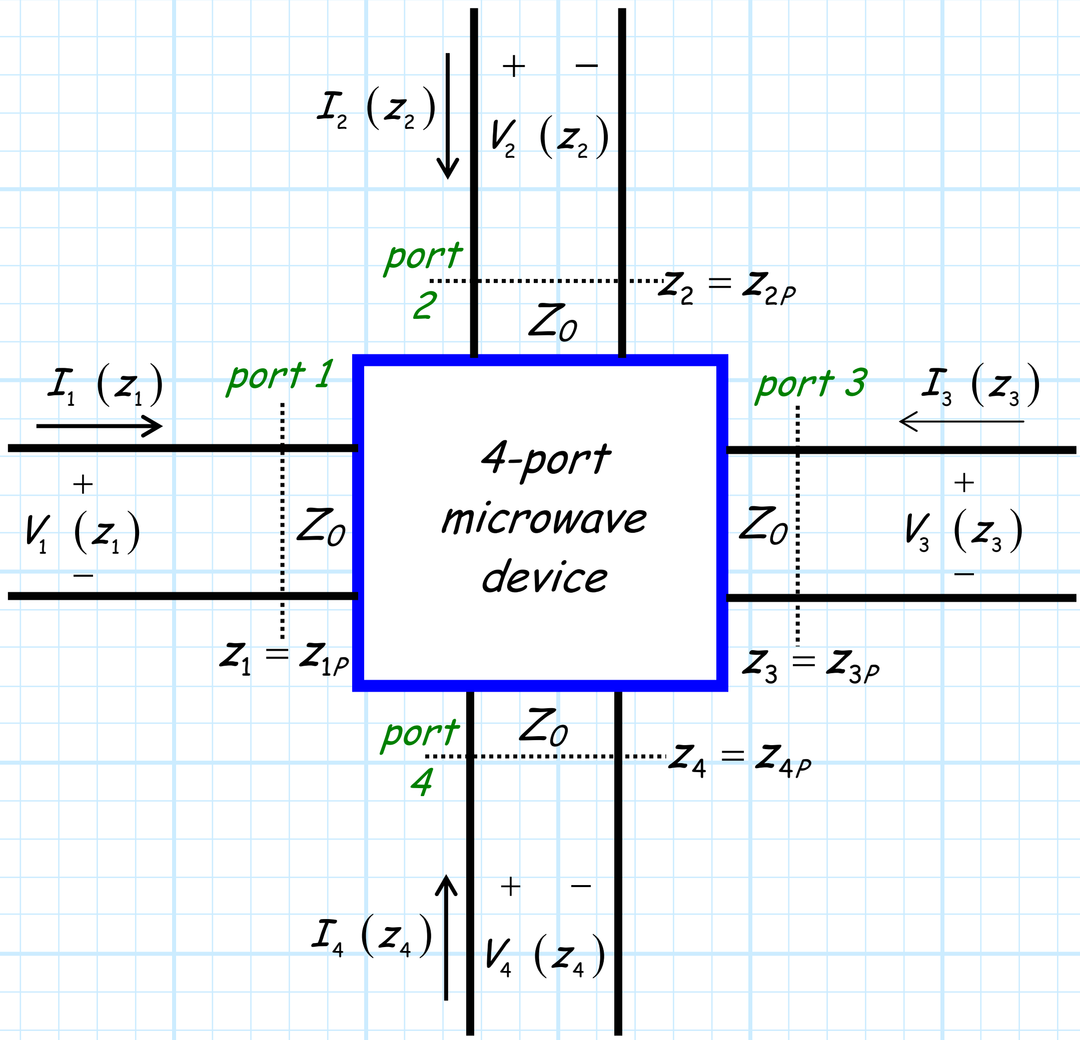
Q: *But, can the scattering matrix by itself tell us anything about the device it characterizes?*

A: Yes! It can tell us if the device is **matched**, or **lossless**, or **reciprocal**.

HO: Matched, Lossless, Reciprocal

The Impedance Matrix

Consider the **4-port** microwave device shown below:



Note in this example, there are four **identical** transmission lines connected to the same "box". Inside this box there may be a very **simple** linear device/circuit, **or** it might contain a very large and **complex** linear microwave system.

→ Either way, the "box" can be fully characterized by its **impedance matrix**!

First, note that each transmission line has a specific **location** that effectively defines the **input** to the device (i.e., z_{1P}, z_{2P}, z_{3P}, z_{4P}). These often arbitrary positions are known as the **port** locations, or port **planes** of the device.

Thus, the **voltage** and **current** at port n is:

$$V_n\left(z_n = z_{nP}\right) \qquad\qquad I_n\left(z_n = z_{nP}\right)$$

We can **simplify** this cumbersome notation by simply **defining** port n current and voltage as I_n and V_n:

$$V_n = V_n\left(z_n = z_{nP}\right) \qquad\qquad I_n = I_n\left(z_n = z_{nP}\right)$$

For **example**, the current at port **3** would be $I_3 = I_3\left(z_3 = z_{3P}\right)$.

Now, say there exists a non-zero current at **port 1** (i.e., $I_1 \neq 0$), while the current at all **other** ports are known to be **zero** (i.e., $I_2 = I_3 = I_4 = 0$).

Say we measure/determine the **current** at port **1** (i.e., determine I_1), and we then measure/determine the **voltage** at the port **2** plane (i.e., determine V_2).

The complex ratio between V_2 and I_1 is know as the **trans-impedance parameter** Z_{21}:

$$Z_{21} = \frac{V_2}{I_1}$$

Likewise, the trans-impedance parameters Z_{31} and Z_{41} are:

$$Z_{31} = \frac{V_3}{I_1} \qquad \text{and} \qquad Z_{41} = \frac{V_4}{I_1}$$

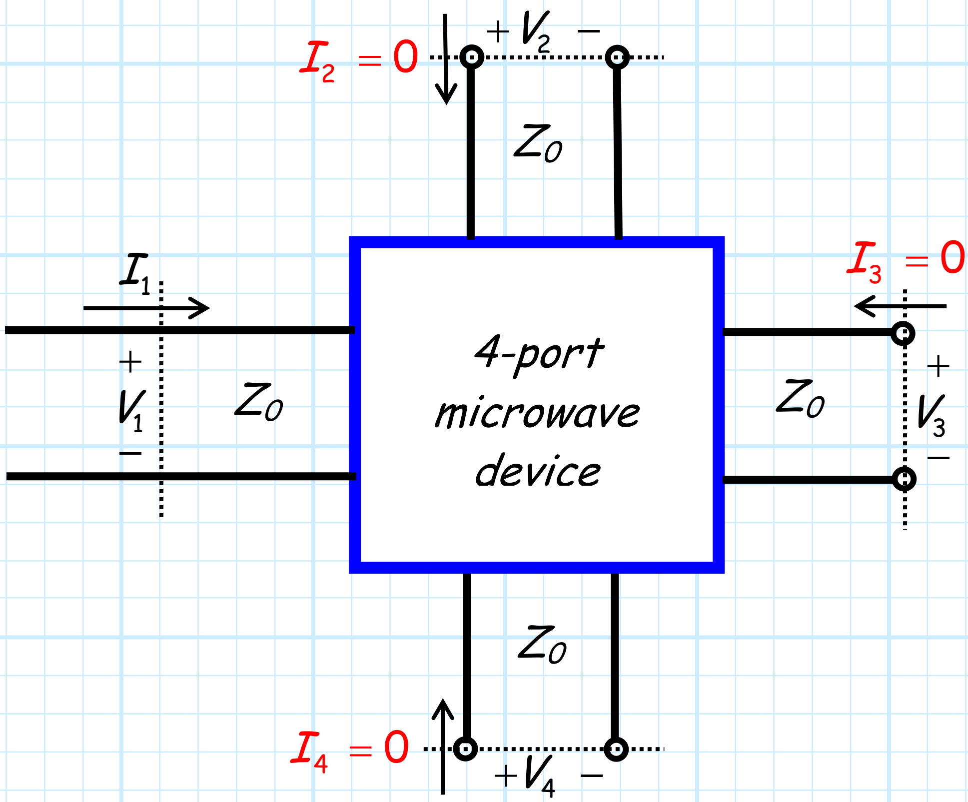
We of course could **also** define, say, trans-impedance parameter Z_{34} as the ratio between the complex values I_4 (the current into port 4) and V_3 (the voltage at port 3), given that the current at all other ports (1, 2, and 3) are zero.

Thus, more **generally**, the ratio of the current into port n and the voltage at port m is:

$$Z_{mn} = \frac{V_m}{I_n} \qquad \text{(given that } I_k = 0 \text{ for all } k \neq n\text{)}$$

Q: *But how do we ensure that all but **one** port current is zero ?*

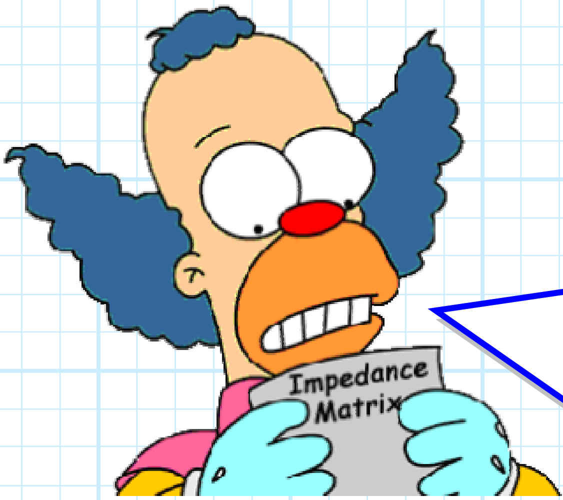
A: Place an **open** circuit **at** those ports!



Placing an **open** at a port (and it must be **at** the port!) **enforces** the condition that $I = 0$.

Now, we can thus **equivalently** state the definition of trans-impedance as:

$$Z_{mn} = \frac{V_m}{I_n} \qquad \text{(given that all ports } k \neq n \text{ are } \textbf{open}\text{)}$$



Q: *As impossible as it sounds, this handout is even more **boring** and **pointless** than any of your previous efforts. **Why** are we studying this? After all, what is the likelihood that a device will have an **open** circuit on **all** but one of its ports?!*

A: OK, say that **none** of our ports are **open-circuited**, such that we have currents **simultaneously** on **each** of the **four** ports of our device.

Since the device is **linear**, the voltage at any **one** port due to **all** the port currents is simply the coherent **sum** of the voltage at that port due to **each** of the currents!

For example, the voltage at port 3 can be determined by:

$$V_3 = Z_{34}\, I_4 + Z_{33}\, I_3 + Z_{32}\, I_2 + Z_{31}\, I_1$$

More generally, the voltage at port m of an N-port device is:

$$V_m = \sum_{n=1}^{N} Z_{mn} I_n$$

This expression can be written in **matrix** form as:

$$\mathbf{V} = \mathcal{Z}\,\mathbf{I}$$

Where $\mathbf{I}$ is the **vector**:

$$\mathbf{I} = \left[I_1, I_2, I_3, \cdots, I_N \right]^T$$

and $\mathbf{V}$ is the vector:

$$\mathbf{V} = \left[V_1, V_2, V_3, \ldots, V_N \right]^T$$

And the matrix $\mathcal{Z}$ is called the **impedance matrix**:

$$\mathcal{Z} = \begin{bmatrix} Z_{11} & \cdots & Z_{1n} \\ \vdots & \ddots & \vdots \\ Z_{m1} & \cdots & Z_{mn} \end{bmatrix}$$

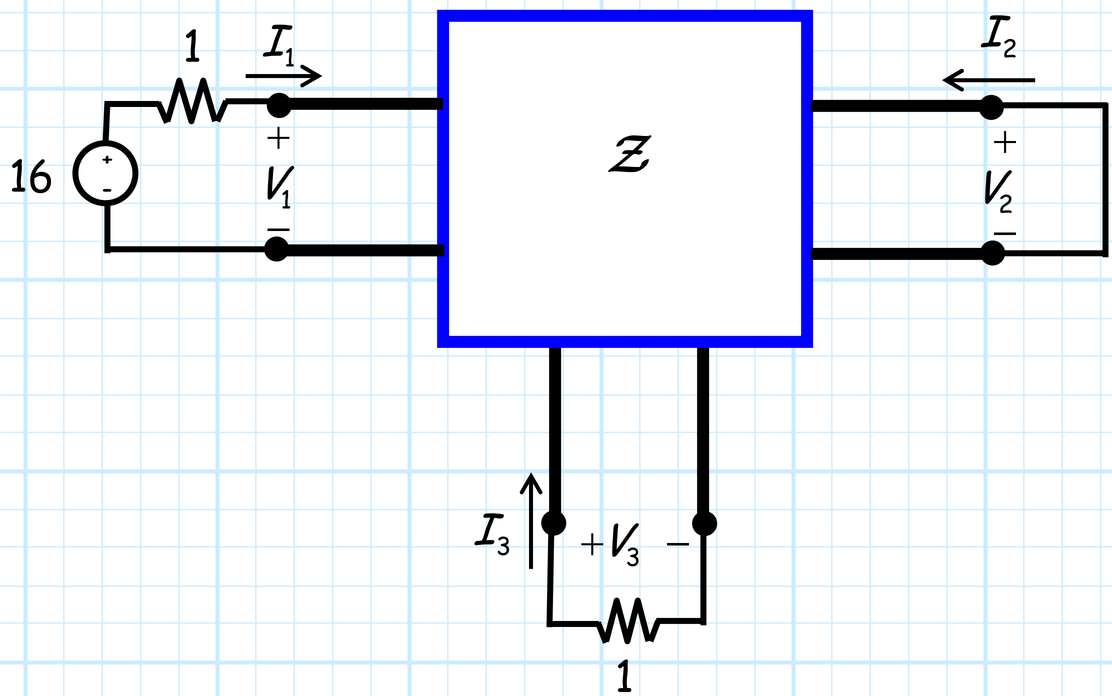
The impedance matrix is a N by N matrix that **completely characterizes** a linear, N-port device. Effectively, the impedance matrix describes a multi-port device the way that Z_L describes a single-port device (e.g., a load)!

But **beware**! The values of the impedance matrix for a particular device or network, just like Z_L, are **frequency dependent**! Thus, it may be more instructive to **explicitly** write:

$$\mathcal{Z}(\omega) = \begin{bmatrix} Z_{11}(\omega) & \ldots & Z_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ Z_{m1}(\omega) & \cdots & Z_{mn}(\omega) \end{bmatrix}$$

Example: Using the Impedance Matrix

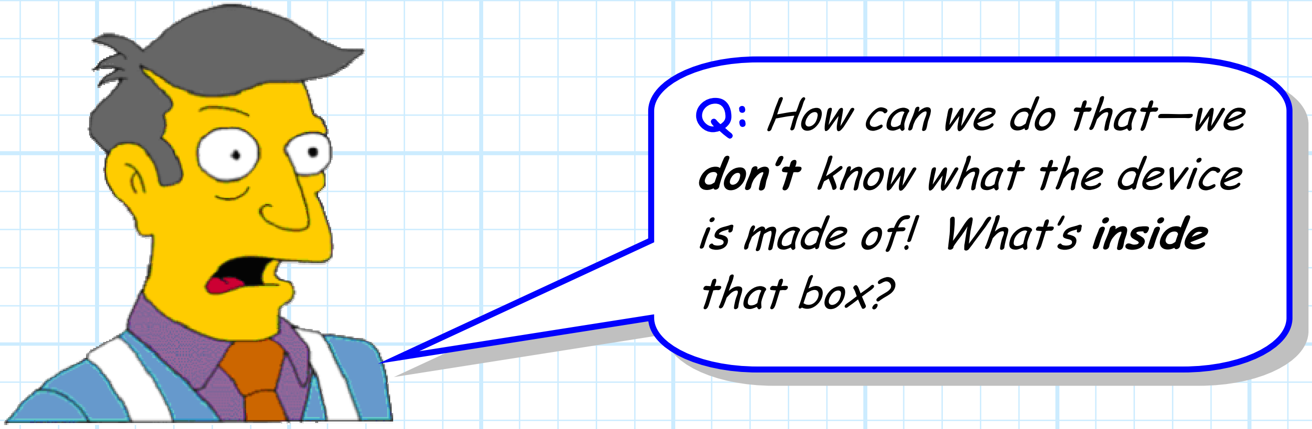
Consider the following circuit:



Where the 3-port **device** is characterized by the **impedance matrix**:

$$\mathcal{Z} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix}$$

Let's now determine all port **voltages** V_1, V_2, V_3 and all **currents** I_1, I_2, I_3.



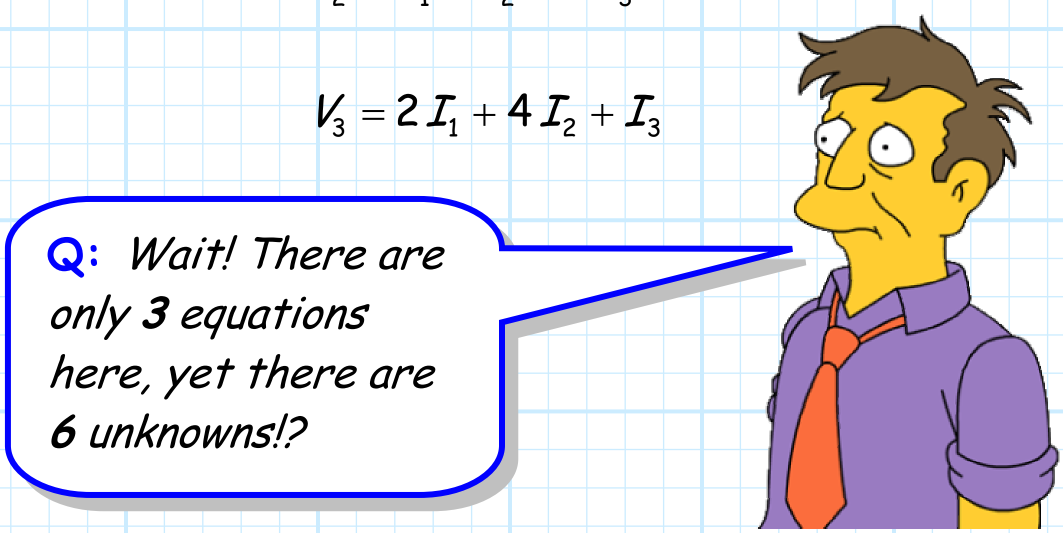
A: We **don't** need to know what's inside that box! We know its impedance matrix, and that **completely** characterizes the device (or, at least, characterizes it at **one** frequency).

Thus, we have enough information to solve this problem. From the impedance matrix we know:

$$V_1 = 2\,I_1 + I_2 + 2\,I_3$$

$$V_2 = I_1 + I_2 + 4\,I_3$$

$$V_3 = 2\,I_1 + 4\,I_2 + I_3$$



A: True! The impedance matrix describes the device in the box, but it does **not** describe the devices **attached** to it. We require **more** equations to describe them.

1. The **source** at port 1 is described by the equation:

$$V_1 = 16.0 - (1)\, I_1$$

2. The **short** circuit on port 2 means that:

$$V_2 = 0$$

3. While the **load** on port 3 leads to:

$$V_3 = -(1)\, I_3 \quad \text{(note the } \textbf{minus sign}\text{!)}$$

Now we have **6** equations and **6** unknowns! Combining equations, we find:

$$V_1 = 16 - I_1 = 2\, I_1 + I_2 + 2\, I_3$$
$$\therefore \quad 16 = 3\, I_1 + I_2 + 2\, I_3$$

$$V_2 = 0 = I_1 + I_2 + 4\, I_3$$
$$\therefore 0 = I_1 + I_2 + 4\, I_3$$

$$V_3 = -I_3 = 2\, I_1 + 4\, I_2 + I_3$$
$$\therefore \quad 0 = 2\, I_1 + 4\, I_2 + 2 I_3$$

Solving, we find (I'll let **you** do the algebraic details!):

$$I_1 = 7.0 \qquad I_2 = -3.0 \qquad I_3 = -1.0$$

$$V_1 = 9.0 \qquad V_2 = 0.0 \qquad V_3 = 1.0$$

The Scattering Matrix

At **"low"** frequencies, we can completely characterize a **linear** device or network using an **impedance** matrix, which relates the currents and voltages at **each** device terminal to the currents and voltages at **all** other terminals.

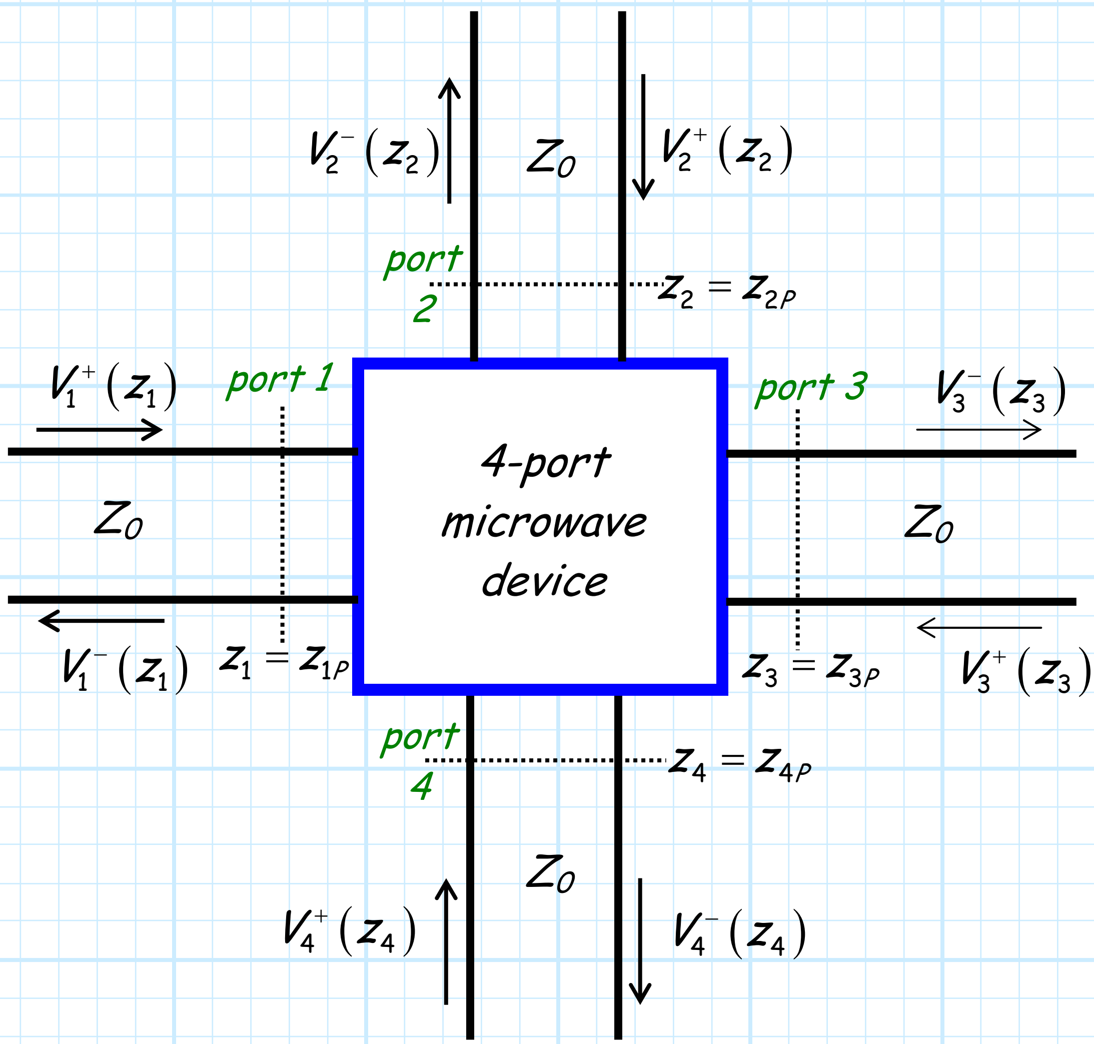
But, at microwave frequencies, it is **difficult** to measure total currents and voltages!

* Instead, we can measure the **magnitude** and **phase** of each of the two transmission line **waves** $V^+(z)$ and $V^-(z)$.

* In other words, we can determine the relationship between the incident and reflected wave at **each** device terminal to the incident and reflected waves at **all** other terminals.

These relationships are completely represented by the **scattering matrix**. It **completely** describes the behavior of a linear, multi-port device at a **given frequency** ω, and a given line impedance Z_0.

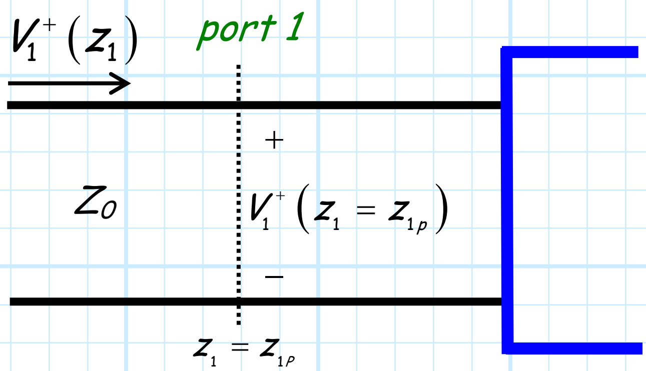
Consider now the **4-port** microwave device shown below:



Note that we have now characterized transmission line activity in terms of incident and "reflected" waves. Note the negative going "reflected" waves can be viewed as the waves **exiting** the multi-port network or device.

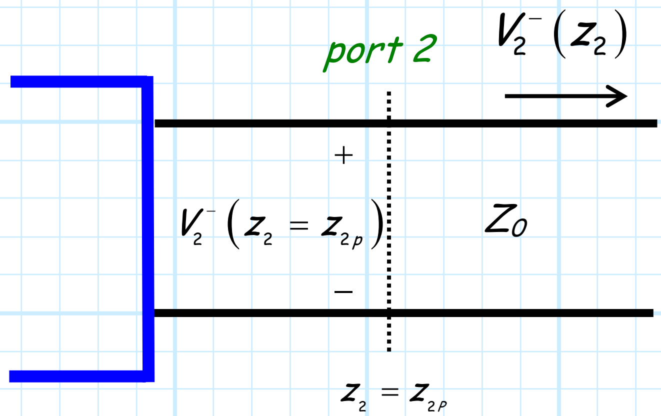
→ Viewing transmission line activity this way, we can fully characterize a multi-port device by its **scattering parameters**!

Say there exists an **incident** wave on **port 1** (i.e., $V_1^+(z_1) \neq 0$), while the incident waves on all other ports are known to be **zero** (i.e., $V_2^+(z_2) = V_3^+(z_3) = V_4^+(z_4) = 0$).



Say we measure/determine the voltage of the wave flowing **into port 1**, at the port 1 **plane** (i.e., determine $V_1^+(z_1 = z_{1P})$).

Say we then measure/determine the voltage of the wave flowing **out** of **port 2**, at the port 2 plane (i.e., determine $V_2^-(z_2 = z_{2P})$).



The complex ratio between $V_1^+(z_1 = z_{1P})$ and $V_2^-(z_2 = z_{2P})$ is know as the **scattering parameter** S_{21}:

$$S_{21} = \frac{V_2^-(z_2 = z_{2P})}{V_1^+(z_1 = z_{1P})} = \frac{V_{02}^- \, e^{+j\beta z_{2P}}}{V_{01}^+ \, e^{-j\beta z_{1P}}} = \frac{V_{02}^-}{V_{01}^+} \quad e^{+j\beta(z_{2P}+z_{1P})}$$

Likewise, the scattering parameters S_{31} and S_{41} are:

$$S_{31} = \frac{V_3^-(z_3 = z_{3P})}{V_1^+(z_1 = z_{1P})} \qquad \text{and} \qquad S_{41} = \frac{V_4^-(z_4 = z_{4P})}{V_1^+(z_1 = z_{1P})}$$

We of course could **also** define, say, scattering parameter S_{34} as the ratio between the complex values $V_4^+(z_4 = z_{4P})$ (the wave **into** port 4) and $V_3^-(z_3 = z_{3P})$ (the wave **out of** port 3), given that the input to all other ports (1,2, and 3) are zero.

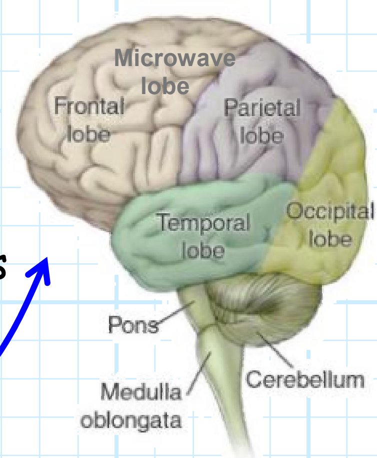
Thus, more **generally**, the ratio of the wave incident on port *n* to the wave emerging from port *m* is:

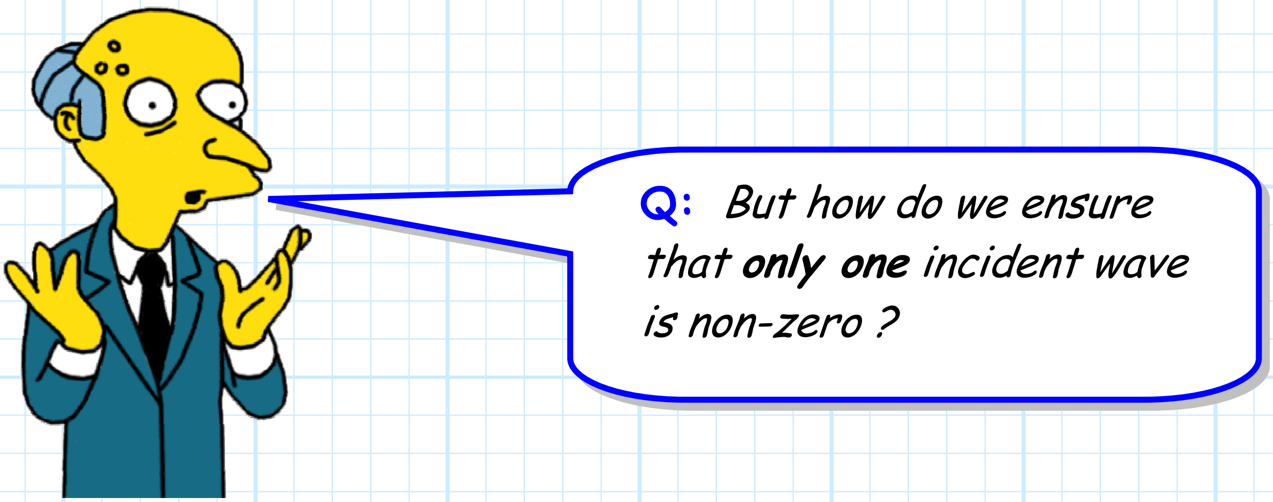
$$S_{mn} = \frac{V_m^-(z_m = z_{mP})}{V_n^+(z_n = z_{nP})} \qquad \text{(given that } \; V_k^+(z_k) = 0 \;\; \text{for all } k \neq n\text{)}$$

Note that frequently the port positions are assigned a **zero** value (e.g., $z_{1P} = 0$, $z_{2P} = 0$). This of course **simplifies** the scattering parameter calculation:

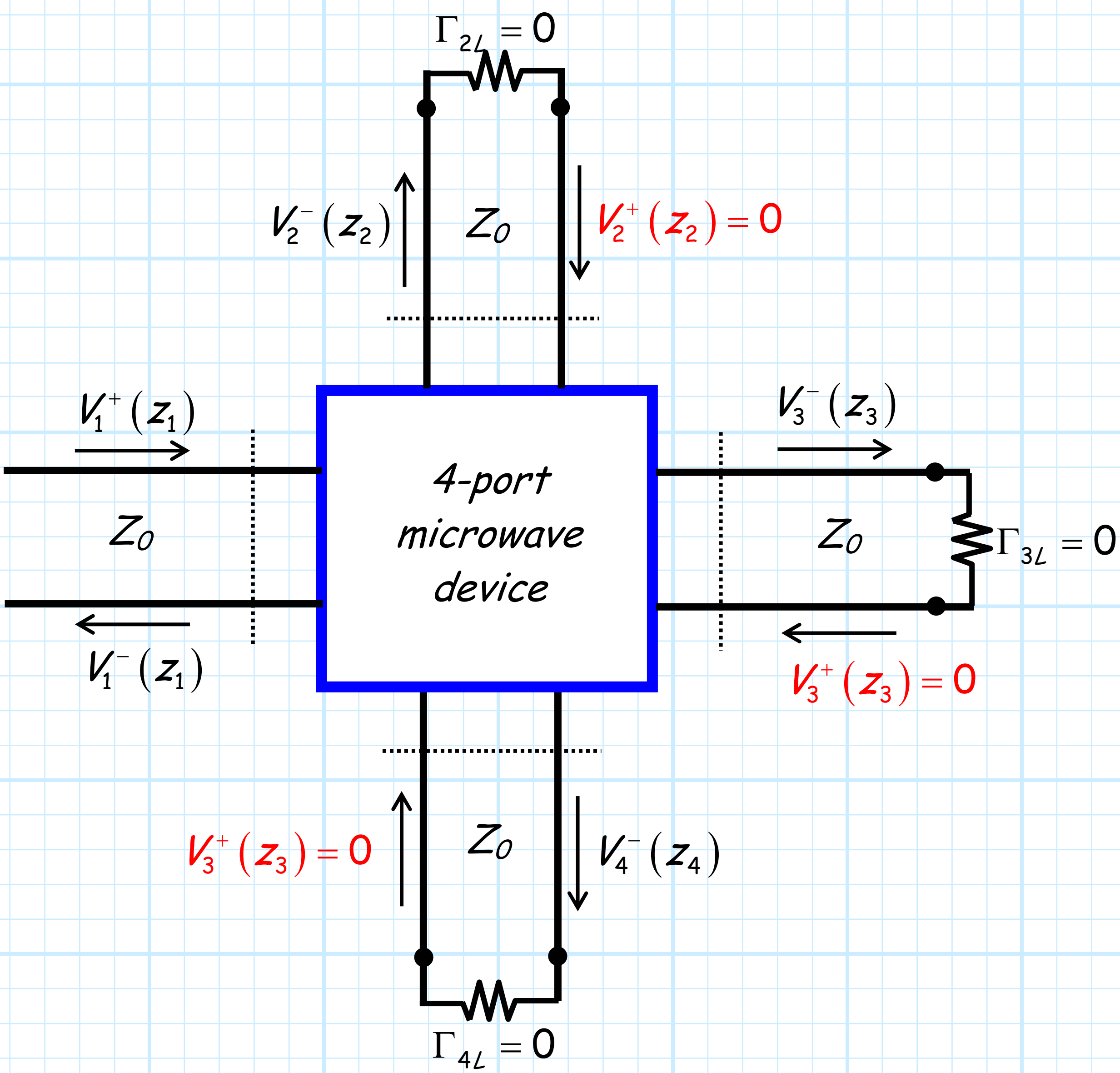
$$S_{mn} = \frac{V_m^-(z_m = 0)}{V_n^+(z_n = 0)} = \frac{V_{0m}^- \, e^{+j\beta 0}}{V_{0n}^+ \, e^{-j\beta 0}} = \frac{V_{0m}^-}{V_{0n}^+}$$

We will **generally assume** that the port locations are defined as $z_{nP} = 0$, and thus use the **above** notation. But **remember** where this expression came from!





A: **Terminate** all other ports with a **matched load**!



Note that **if** the ports are terminated in a **matched load** (i.e., $Z_L = Z_0$), then $\Gamma_{nL} = 0$ and therefore:

$$V_n^+(z_n) = 0$$

In other words, terminating a port ensures that there will be **no signal** incident on that port!

Q: *Just between you and me, I think you've messed this up!* ***In all*** *previous handouts you said that if* $\Gamma_L = 0$*, the wave in the* ***minus*** *direction would be zero:*

$$V^-(z) = 0 \quad \text{if} \quad \Gamma_L = 0$$

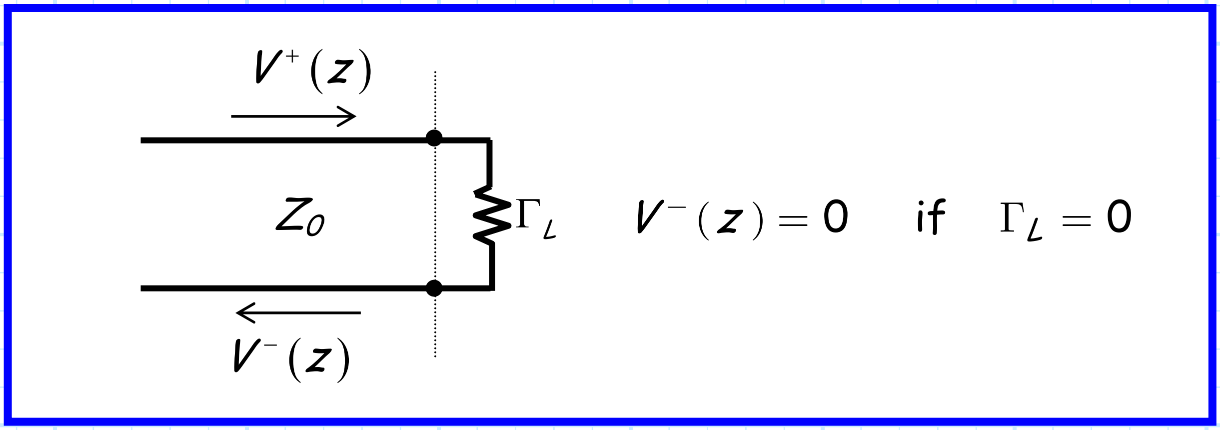
but just ***now*** *you said that the wave in the* ***positive*** *direction would be zero:*

$$V^+(z) = 0 \quad \text{if} \quad \Gamma_L = 0$$

Of course, there is ***no way*** *that* ***both*** *statements can be correct!*

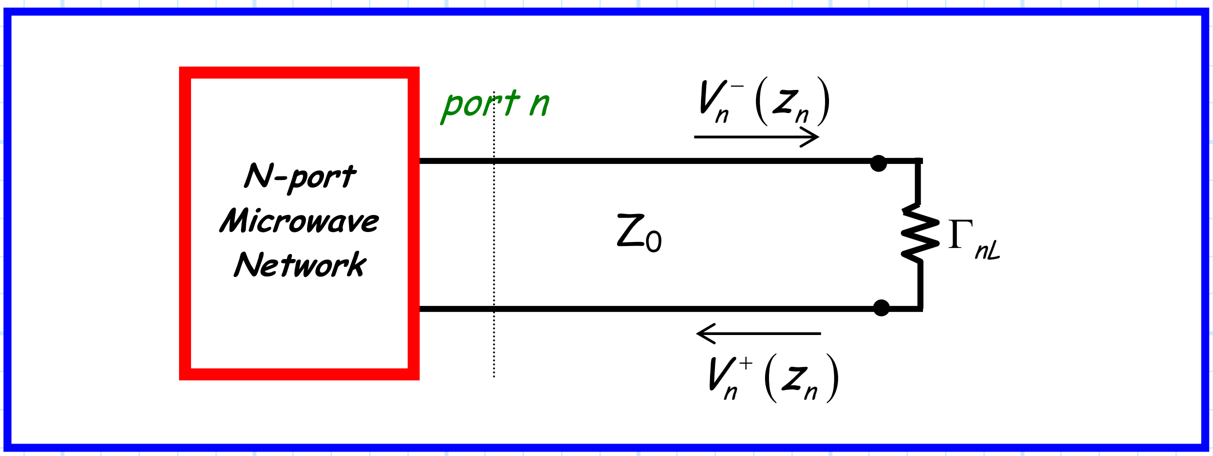
A: Actually, **both** statements are correct! You must be careful to understand the **physical definitions** of the plus and minus directions—in other words, the propagation directions of waves $V_n^+(z_n)$ and $V_n^-(z_n)$!

For example, we **originally** analyzed this case:



In this original case, the wave **incident** on the load is $V^+(z)$ (**plus** direction), while the **reflected** wave is $V^-(z)$ (**minus** direction).

Contrast this with the case we are **now** considering:

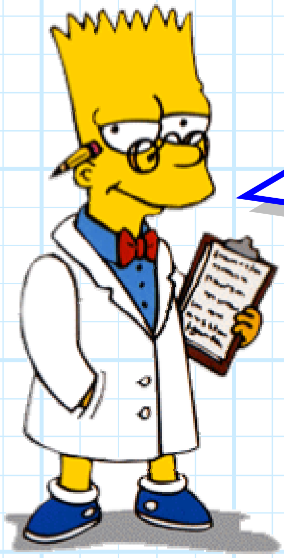


For this current case, the situation is **reversed**. The wave incident on the load is **now** denoted as $V_n^-(z_n)$ (coming **out** of port *n*), while the wave reflected off the load is **now** denoted as $V_n^+(z_n)$ (going **into** port *n*).

As a result, $V_n^+(z_n) = 0$ when $\Gamma_{nL} = 0$!

Perhaps we could more **generally** state that for some load Γ_L:

$$V^{reflected}(z = z_L) = \Gamma_L \, V^{incident}(z = z_L)$$



*For each case, **you** must be able to correctly identify the mathematical statement describing the wave **incident** on, and **reflected** from, some passive load.*

*Like most equations in engineering, the **variable names** can **change**, but the **physics** described by the mathematics will **not**!*

Now, **back** to our discussion of **S-parameters**. We found that **if** $z_{nP} = 0$ for all ports n, the scattering parameters could be directly written in terms of wave **amplitudes** V_{0n}^{+} and V_{0m}^{-}.

$$S_{mn} = \frac{V_{0m}^{-}}{V_{0n}^{+}} \qquad \text{(when } V_k^{+}(z_k) = 0 \text{ for all } k \neq n\text{)}$$

Which we can now **equivalently** state as:

$$S_{mn} = \frac{V_{0m}^{-}}{V_{0n}^{+}} \qquad \text{(when all ports, except port } n\text{, are terminated in \textbf{matched loads})}$$

We can use the scattering matrix to determine the solution for a more **general** circuit—one where the ports are **not** terminated in matched loads!

Q: *I'm not understanding the importance scattering parameters. How are they useful to us **microwave engineers**?*

A: Since the device is **linear**, we can apply **superposition**. The output at any port due to **all** the incident waves is simply the coherent **sum** of the output at that port due to **each** wave!

For example, the **output** wave at port 3 can be determined by (assuming $z_{nP} = 0$):

$$V_{03}^{-} = S_{34} V_{04}^{+} + S_{33} V_{03}^{+} + S_{32} V_{02}^{+} + S_{31} V_{01}^{+}$$

More **generally**, the output at port m of an N-port device is:

$$V_{0m}^{-} = \sum_{n=1}^{N} S_{mn} V_{0n}^{+} \qquad (z_{nP} = 0)$$

This expression can be written in **matrix** form as:

$$\mathbf{V}^{-} = \mathcal{S}\ \mathbf{V}^{+}$$

Where $\mathbf{V}^-$ is the **vector**:

$$\mathbf{V}^- = \left[V_{01}^-, V_{02}^-, V_{03}^-, \ldots, V_{0N}^-\right]^T$$

and $\mathbf{V}^+$ is the vector:

$$\mathbf{V}^+ = \left[V_{01}^+, V_{02}^+, V_{03}^+, \ldots, V_{0N}^+\right]^T$$

Therefore $\mathcal{S}$ is the **scattering matrix**:

$$\mathcal{S} = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{m1} & \cdots & S_{mn} \end{bmatrix}$$

The scattering matrix is a N by N matrix that **completely characterizes** a linear, N-port device. Effectively, the scattering matrix describes a multi-port device the way that Γ_L describes a single-port device (e.g., a load)!

But **beware**! The values of the scattering matrix for a particular device or network, just like Γ_L, are **frequency dependent**! Thus, it may be more instructive to **explicitly** write:

$$\mathcal{S}(\omega) = \begin{bmatrix} \mathcal{S}_{11}(\omega) & \ldots & \mathcal{S}_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ \mathcal{S}_{m1}(\omega) & \cdots & \mathcal{S}_{mn}(\omega) \end{bmatrix}$$

Also realize that—also just like Γ_L—the scattering matrix is dependent on **both** the **device/network** and the Z_0 value of the **transmission lines connected** to it.

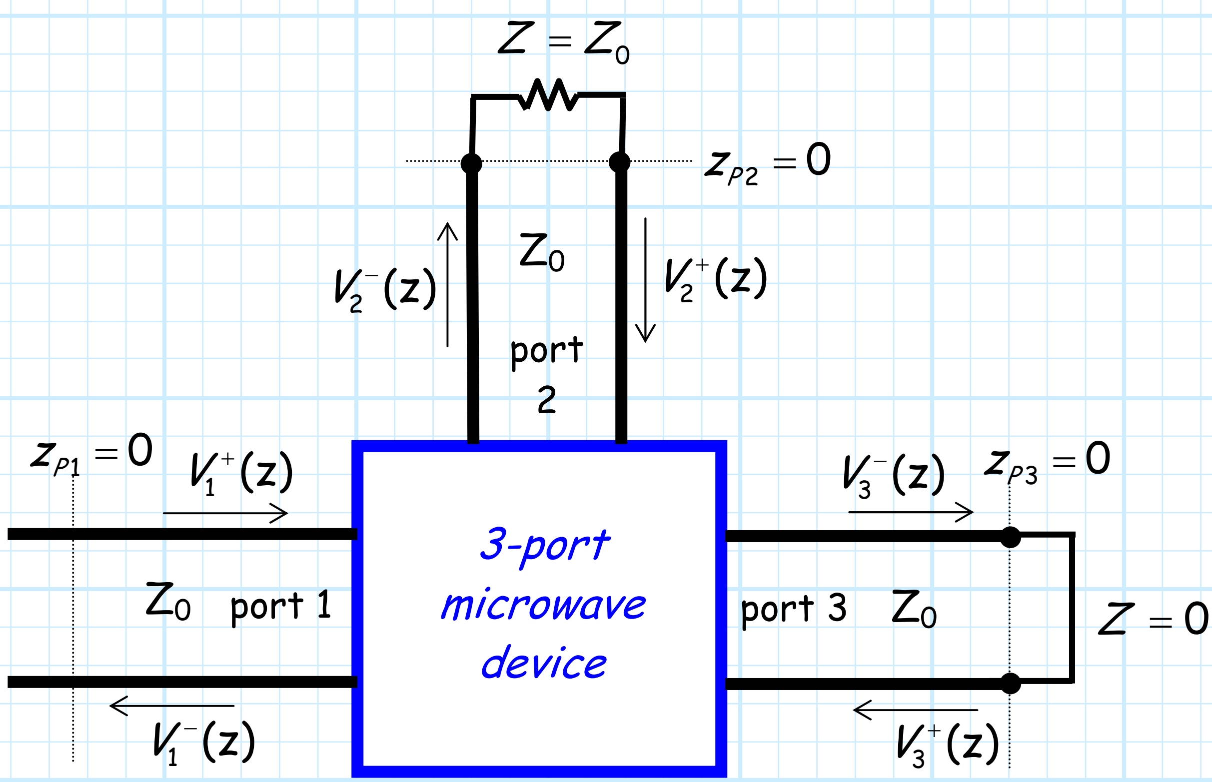
Thus, a device connected to transmission lines with $Z_0 = 50\Omega$ will have a **completely different scattering matrix** than that same device connected to transmission lines with $Z_0 = 100\Omega$!!!

Example: The Scattering Matrix

Say we have a 3-port network that is completely characterized at some frequency ω by the **scattering matrix**:

$$\mathcal{S} = \begin{bmatrix} 0.0 & 0.2 & 0.5 \\ 0.5 & 0.0 & 0.2 \\ 0.5 & 0.5 & 0.0 \end{bmatrix}$$

A **matched load** is attached to port 2, while a **short circuit** has been placed at port 3:



a) Find the **reflection** coefficient at port 1, i.e.:

$$\Gamma_1 \doteq \frac{V_{01}^-}{V_{01}^+}$$

b) Find the **transmission** coefficient from port 1 to port 2, i.e.,

$$T_{21} \doteq \frac{V_{02}^-}{V_{01}^+}$$

*I am amused by the trivial problems that **you** apparently find so difficult. **I** know that:*

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} = S_{11} = 0.0$$

and

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} = S_{21} = 0.5$$

NO!!! The above statement is **not correct**!

Remember, $V_{01}^- / V_{01}^+ = S_{11}$ **only** if ports 2 and 3 are terminated in **matched** loads! In this problem port 3 is terminated with a **short circuit**.

Therefore:

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} \neq S_{11}$$

and similarly:

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} \neq S_{21}$$

To determine the values T_{21} and Γ_1, we must start with the **three** equations provided by the **scattering matrix**:

$$V_{01}^- = \qquad\qquad 0.2\, V_{02}^+ + 0.5\, V_{03}^+$$

$$V_{02}^- = 0.5\, V_{01}^+ \qquad\qquad + 0.2\, V_{03}^+$$

$$V_{03}^- = 0.5\, V_{01}^+ + 0.5\, V_{02}^+$$

and the **two** equations provided by the **attached loads**:

$$\Gamma_{L2} = 0 \quad \Rightarrow \quad V_{02}^+ = 0$$

$$\Gamma_{L3} = -1 \quad \Rightarrow \quad V_{03}^+ = -V_{03}^-$$

You've made a terrible mistake! Fortunately, ***I*** *was here to correct it for you—since $\Gamma_L = 0$, the constant V_{02}^-* ***(not V_{02}^+)*** *is equal to zero.*

NO!! Remember, the signal $V_2^-(z)$ is **incident** on the matched load, and $V_2^+(z)$ is the **reflected** wave from the load (i.e., $V_2^+(z)$ is incident on **port 2**). Therefore, $V_{02}^+=0$ **is correct**!

Likewise, because of the **short** circuit at port 3 ($\Gamma_L=-1$):

$$\frac{V_3^+(z_3=0)}{V_3^-(z_3=0)}=\frac{V_{03}^+}{V_{03}^-}=-1$$

and therefore:

$$V_{03}^+=-V_{03}^-$$

We can divide all of these equations by V_{01}^+, resulting in:

$$\Gamma_1=\frac{V_{01}^-}{V_{01}^+}= \qquad 0.2\,\frac{V_{02}^+}{V_{01}^+}+0.5\,\frac{V_{03}^+}{V_{01}^+}$$

$$T_{21}=\frac{V_{02}^-}{V_{01}^+}=0.5 \qquad +0.2\,\frac{V_{03}^+}{V_{01}^+}$$

$$\frac{V_{03}^-}{V_{01}^+}=0.5+0.5\,\frac{V_{02}^+}{V_{01}^+}$$

$$\frac{V_{02}^+}{V_{01}^+}=0$$

$$\frac{V_{03}^+}{V_{01}^+}=-\frac{V_{03}^-}{V_{01}^+}$$

Look what we have—**5** equations and **5** unknowns! Inserting equations 4 and 5 into equations 1 through 3, we get:

$$\Gamma_1 = \frac{V_{01}^-}{V_{01}^+} = -0.5\,\frac{V_{03}^+}{V_{01}^+}$$

$$T_{21} = \frac{V_{02}^-}{V_{01}^+} = 0.5 - 0.2\,\frac{V_{03}^+}{V_{01}^+}$$

$$\frac{V_{03}^-}{V_{01}^+} = 0.5$$

Solving, we find:

$$\Gamma_1 = -0.5(0.5) = -0.25$$

$$T_{21} = 0.5 - 0.2(0.5) = 0.4$$

Example: Scattering Parameters

Consider a **two-port device** with a scattering matrix (at some specific frequency ω_0):

$$\mathcal{S}(\omega=\omega_0)=\begin{bmatrix} 0.1 & j0.7 \\ j0.7 & -0.2 \end{bmatrix}$$

and $Z_0 = 50\Omega$.

Say that the transmission line connected to **port 2** of this device is terminated in a **matched** load, and that the wave **incident** on **port 1** is:

$$V_1^+(z_1) = -j2\, e^{-j\beta z_1}$$

where $z_{1P} = z_{2P} = 0$.

Determine:

1. the port voltages $V_1(z_1 = z_{1P})$ and $V_2(z_2 = z_{2P})$.

2. the port currents $I_1(z_1 = z_{1P})$ and $I_2(z_2 = z_{2P})$.

3. the net power flowing into port 1

1. Since the **incident** wave on port 1 is:

$$V_1^+(z_1) = -j2\, e^{-j\beta z_1}$$

we can conclude (since $z_{1P} = 0$):

$$\begin{aligned} V_1^+(z_1 = z_{1P}) &= -j2\, e^{-j\beta z_{1P}} \\ &= -j2\, e^{-j\beta(0)} \\ &= -j2 \end{aligned}$$

and since port 2 is **matched** (and **only** because its matched!), we find:

$$\begin{aligned} V_1^-(z_1 = z_{1P}) &= S_{11}\, V_1^+(z_1 = z_{1P}) \\ &= 0.1(-j2) \\ &= -j0.2 \end{aligned}$$

The voltage at port 1 is thus:

$$\begin{aligned} V_1(z_1 = z_{1P}) &= V_1^+(z_1 = z_{1P}) + V_1^-(z_1 = z_{1P}) \\ &= -j2.0 - j0.2 \\ &= -j2.2 \\ &= 2.2\, e^{-j\pi/2} \end{aligned}$$

Likewise, since port 2 is **matched**:

$$V_2^+(z_2 = z_{2P}) = 0$$

And also:

$$
\begin{aligned}
V_2^-(z_2 = z_{2P}) &= S_{21}\, V_1^+(z_1 = z_{1P}) \\
&= j0.7\,(-j2) \\
&= 1.4
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
V_2(z_2 = z_{2P}) &= V_2^+(z_2 = z_{2P}) + V_2^-(z_2 = z_{2P}) \\
&= 0 + 1.4 \\
&= 1.4 \\
&= 1.4\, e^{-j0}
\end{aligned}
$$

2. The port **currents** can be easily determined from the results of the previous section.

$$
\begin{aligned}
I_1(z_1 = z_{1P}) &= I_1^+(z_1 = z_{1P}) - I_1^-(z_1 = z_{1P}) \\
&= \frac{V_1^+(z_1 = z_{1P})}{Z_0} - \frac{V_1^-(z_1 = z_{1P})}{Z_0} \\
&= -j\frac{2.0}{50} + j\frac{0.2}{50} \\
&= -j\frac{1.8}{50} \\
&= -j0.036 \\
&= 0.036\, e^{-j\pi/2}
\end{aligned}
$$

and:

$$
\begin{aligned}
I_2(z_2 = z_{2P}) &= I_2^+(z_2 = z_{2P}) - I_2^-(z_2 = z_{2P}) \\
&= \frac{V_2^+(z_2 = z_{2P})}{Z_0} - \frac{V_2^-(z_2 = z_{2P})}{Z_0} \\
&= \frac{0}{50} - \frac{1.4}{50} \\
&= -0.028 \\
&= 0.028\, e^{+j\pi}
\end{aligned}
$$

3. The **net power** flowing into port 1 is:

$$
\begin{aligned}
\Delta P_1 &= P_1^+ - P_1^- \\
&= \frac{\left|V_{01}^+\right|^2}{2Z_0} - \frac{\left|V_{01}^-\right|^2}{2Z_0} \\
&= \frac{(2)^2 - (0.2)^2}{2(50)} \\
&= 0.0396 \quad \textit{Watts}
\end{aligned}
$$

Matched, Lossless, Reciprocal Devices

A microwave device can be **lossless** or **reciprocal**. In addition, we can likewise classify it as being **matched**.

Let's examine **each** of these three characteristics, and how they relate to the **scattering matrix**.

Matched

A matched device is another way of saying that the **input impedance** at each port is **equal to Z_0** when **all other** ports are terminated in **matched loads**. For this condition, the **reflection coefficient** of each port is **zero**—no signal will be come out of a port if a signal is incident on that port (but **only** that port!).

In other words, we want:

$$V_m^- = S_{mm} V_m^+ = 0 \quad \text{for all } m$$

a result that occurs when:

$$S_{mm} = 0 \quad \text{for all } m \text{ if matched}$$

We find therefore that a matched device will exhibit a scattering matrix where all **diagonal elements** are **zero**.

Therefore:

$$\mathcal{S} = \begin{bmatrix} 0 & 0.1 & j0.2 \\ 0.1 & 0 & 0.3 \\ j0.2 & 0.3 & 0 \end{bmatrix}$$

is an example of a scattering matrix for a **matched,** three port device.

Lossless

For a lossless device, all of the power that delivered to each device port must eventually find its way **out**!

In other words, power is not **absorbed** by the network—no power to be **converted to heat**!

Recall the **power incident** on some port m is related to the amplitude of the **incident wave** (V_{0m}^{+}) as:

$$P_m^{+} = \frac{\left|V_{0m}^{+}\right|^2}{2Z_0}$$

While power of the **wave exiting** the port is:

$$P_m^{-} = \frac{\left|V_{0m}^{-}\right|^2}{2Z_0}$$

Thus, the power **delivered** to (absorbed by) that port is the **difference** of the two:

$$\Delta P_m = P_m^+ - P_m^- = \frac{\left|V_{0m}^+\right|^2}{2Z_0} - \frac{\left|V_{0m}^-\right|^2}{2Z_0}$$

Thus, the **total power incident** on an N-port device is:

$$P^+ = \sum_{m=1}^{N} P_m^+ = \frac{1}{2Z_0}\sum_{m=1}^{N}\left|V_{0m}^+\right|^2$$

Note that:

$$\sum_{m=1}^{N}\left|V_{0m}^+\right|^2 = \left(\mathbf{V}^+\right)^H \mathbf{V}^+$$

where operator H indicates the **conjugate transpose** (i.e., Hermetian transpose) operation, so that $\left(\mathbf{V}^+\right)^H \mathbf{V}^+$ is the **inner product** (i.e., dot product, or scalar product) of complex vector $\mathbf{V}^+$ with itself.

Thus, we can write the **total power incident** on the device as:

$$P^+ = \frac{1}{2Z_0}\sum_{m=1}^{N}\left|V_{0m}^+\right|^2 = \frac{\left(\mathbf{V}^+\right)^H \mathbf{V}^+}{2Z_0}$$

Similarly, we can express the **total power** of the **waves exiting** our M-port network to be:

$$P^- = \frac{1}{2Z_0}\sum_{m=1}^{N}\left|V_{0m}^-\right|^2 = \frac{\left(\mathbf{V}^-\right)^H \mathbf{V}^-}{2Z_0}$$

Now, recalling that the incident and exiting wave amplitudes are **related** by the **scattering matrix** of the device:

$$\mathbf{V}^- = \mathcal{S}\ \mathbf{V}^+$$

Thus we find:

$$P^- = \frac{\left(\mathbf{V}^-\right)^H \mathbf{V}^-}{2Z_0} = \frac{\left(\mathbf{V}^+\right)^H \mathcal{S}^H \mathcal{S}\ \mathbf{V}^+}{2Z_0}$$

Now, the **total power delivered** to the network is:

$$\Delta P = \sum_{m=1}^{M} \Delta P = P^+ - P^-$$

Or explicitly:

$$\begin{aligned}\Delta P &= P^+ - P^- \\ &= \frac{\left(\mathbf{V}^+\right)^H \mathbf{V}^+}{2Z_0} - \frac{\left(\mathbf{V}^+\right)^H \mathcal{S}^H \mathcal{S}\ \mathbf{V}^+}{2Z_0} \\ &= \frac{1}{2Z_0}\ \left(\mathbf{V}^+\right)^H \left(\mathcal{I} - \mathcal{S}^H \mathcal{S}\right) \mathbf{V}^+\end{aligned}$$

where $\mathcal{I}$ is the **identity matrix**.

Q: *Is there actually some* ***point*** *to this long, rambling, complex presentation?*

A: Absolutely! If our M-port device is lossless then the total power exiting the device must **always** be equal to the total power incident on it.

If network is **lossless,** then $P^+ = P^-$.

Or stated another way, the total **power delivered** to the device (i.e., the power absorbed by the device) must always be **zero** if the device is lossless!

If network is **lossless,** then $\Delta P = 0$

Thus, we can conclude from our math that for a **lossless device**:

$$\Delta P = \frac{1}{2Z_0} \left(\mathbf{V}^+\right)^H \left(\mathcal{I} - \mathcal{S}^H \mathcal{S}\right) \mathbf{V}^+ = 0 \qquad \text{for all } \mathbf{V}^+$$

This is true **only** if:

$$\mathcal{I} - \mathcal{S}^H \mathcal{S} = 0 \qquad \Rightarrow \qquad \mathcal{S}^H \mathcal{S} = \mathcal{I}$$

Thus, we can conclude that the **scattering matrix** of a **lossless** device has the **characteristic**:

If a network is **lossless**, then $\mathcal{S}^H \mathcal{S} = \mathcal{I}$

Q: *Huh? What exactly is this supposed to* ***tell*** *us?*

A: A matrix that satisfies $\mathcal{S}^H \mathcal{S} = \mathcal{I}$ is a special kind of matrix known as a **unitary matrix**.

If a network is lossless, then its scattering matrix $\mathcal{S}$ is unitary.

Q: *How do I **recognize** a unitary matrix if I **see** one?*

A: The **columns** of a unitary matrix form an **orthonormal set**!

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{22} \\ S_{31} & S_{33} & S_{33} & S_{33} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}$$

matrix columns

In other words, each **column** of the scattering matrix will have a **magnitude equal to one**:

$$\sum_{m=1}^{N} \left| S_{mn} \right|^2 = 1 \quad \text{for all } n$$

while the inner product (i.e., dot product) of **dissimilar columns** must be **zero**.

$$\sum_{n=1}^{N} S_{ni} S_{nj}^* = S_{1i} S_{1j}^* + S_{2i} S_{2j}^* + \cdots + S_{Ni} S_{Nj}^* = 0 \quad \text{for all } i \neq j$$

In other words, dissimilar columns are **orthogonal**.

Consider, for example, a lossless **three-port** device. Say a signal is incident on port 1, and that **all** other ports are **terminated**. The power **incident** on port 1 is therefore:

$$P_1^+ = \frac{|V_{01}^+|^2}{2Z_0}$$

while the power **exiting** the device at each port is:

$$P_m^- = \frac{|V_{0m}^-|^2}{2Z_0} = \frac{|S_{m1}V_{01}^-|^2}{2Z_0} = |S_{m1}|^2 P_1^+$$

The **total** power exiting the device is therefore:

$$\begin{aligned} P^- &= P_1^- + P_2^- + P_3^- \\ &= |S_{11}|^2 P_1^+ + |S_{21}|^2 P_1^+ + |S_{31}|^2 P_1^+ \\ &= \left(|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2\right) P_1^+ \end{aligned}$$

Since this device is **lossless**, then the incident power (**only** on port 1) is **equal** to exiting power (i.e, $P^- = P_1^+$). This is true **only** if:

$$|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 = 1$$

Of course, this will likewise be true if the incident wave is placed on **any** of the **other** ports of this lossless device:

$$|S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2 = 1$$
$$|S_{13}|^2 + |S_{23}|^2 + |S_{33}|^2 = 1$$

We can state in general then that:

$$\sum_{m=1}^{3} \left|S_{mn}\right|^2 = 1 \quad \text{for all } n$$

In other words, the columns of the scattering matrix must have **unit magnitude** (a requirement of all **unitary** matrices). It is apparent that this must be true for energy to be conserved.

An **example** of a (unitary) scattering matrix for a **lossless** device is:

$$\mathcal{S} = \begin{bmatrix} 0 & \tfrac{1}{2} & j\,\tfrac{\sqrt{3}}{2} & 0 \\ \tfrac{1}{2} & 0 & 0 & j\,\tfrac{\sqrt{3}}{2} \\ j\,\tfrac{\sqrt{3}}{2} & 0 & 0 & \tfrac{1}{2} \\ 0 & j\,\tfrac{\sqrt{3}}{2} & \tfrac{1}{2} & 0 \end{bmatrix}$$

Reciprocal

Recall **reciprocity** results when we build a **passive** (i.e., unpowered) device with **simple** materials.

For a reciprocal network, we find that the elements of the scattering matrix are **related** as:

$$S_{mn} = S_{nm}$$

For example, a **reciprocal** device will have $S_{21} = S_{12}$ or $S_{32} = S_{23}$. We can write reciprocity in matrix form as:

$$\mathcal{S}^T = \mathcal{S} \qquad \text{if reciprocal}$$

where T indicates (non-conjugate) transpose.

An **example** of a scattering matrix describing a **reciprocal**, but **lossy** and **non-matched** device is:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} 0.10 & -0.40 & -j0.20 & 0.05 \\ -0.40 & j0.20 & 0 & j0.10 \\ -j0.20 & 0 & 0.10 - j0.30 & -0.12 \\ 0.05 & j0.10 & -0.12 & 0 \end{bmatrix}$$