B. Spherical Wave Propagation

Every antenna launches a **spherical wave**, thus its power density reduces as a function of $1/r^2$, where r is the **distance** from the antenna.

We can determine the **power** radiated by an antenna if we know the **power density** of the propagating spherical wave it produces.

HO: Total Radiated Power

To **describe** how an antenna **distributes** radiated power, we first need to understand the concept of a **solid angle**—measured in units of **steradians**.

HO: The Steradian

We find that an antenna **never** radiates power **equally** in all directions. Instead, it radiates power more in some directions and less in others. The mathematical description of this distribution is called **radiation intensity**.

HO: Radiation Intensity

Total Radiated Power

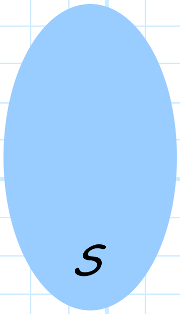
So, we know that an **antenna** (located at the origin) will produce a **radiated power density** of the form:

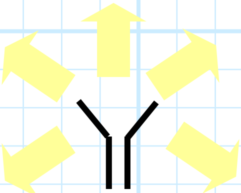
$$\mathbf{W}(\bar{r}) = U(\theta,\phi)\frac{\hat{\mathbf{r}}}{r^2}$$

Q: *But this is the power **density**—a **function** of spatial position (i.e., a function of r,θ,ϕ). Is there any way to determine the **total power radiated** by an antenna?*

A: The power density function gives us **all** we need to know to determine the power radiated by an antenna (P_{rad}).

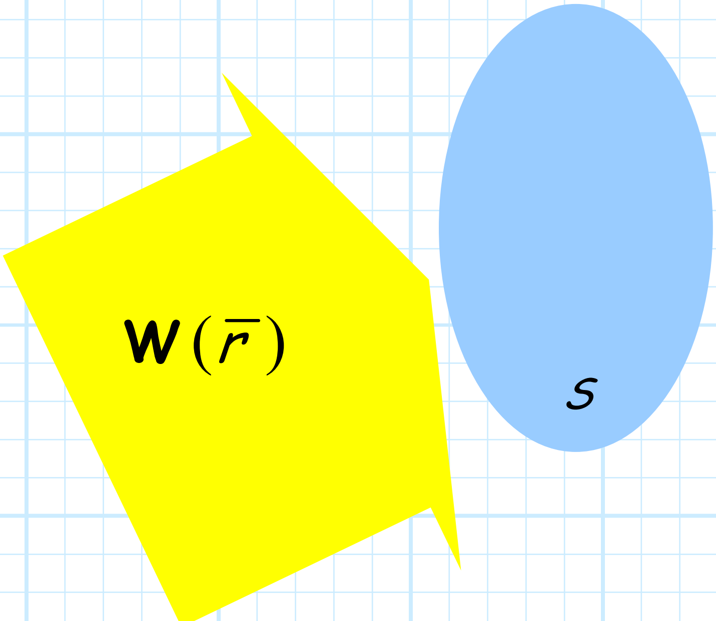
To see why, first consider some **aperture** (i.e., window) defined by **surface area** S. Say that at some (perhaps large) distance away from this surface, there exists a radiating **antenna**.





This radiating antenna produces a propagating electromagnetic wave at **all** points throughout the **entire universe**!

The entire universe **includes** our aperture (surface S) and thus there **must** be electromagnetic energy propagating **through** this aperture (i.e., from one side of surface S to the other).



Q: *But an electromagnetic wave contains* ***energy****, and thus energy must be passing* ***through*** *this aperture. Can we determine the* ***rate*** *of this energy flow through surface S?*

A: No problem! If we know the power density $\mathbf{W}(\bar{r})$, we can always determine the **power** flowing through some aperture S using a **surface integration**:

$$P = \iint_S \mathbf{W}(\bar{r}) \cdot \overline{ds}$$

Hopefully this makes sense to **you**! We simply integrate the power **density** (in ***W/m***^{2}) flowing through the surface S (in ***m***^{2}) to determine the **rate** of energy flow (in **Watts**) through the entire surface S.

Q: *So the value P is the power radiated by the* ***antenna*** *?*

A: **Absolutely not!** Although the power P does depend on the power density produced by the antenna, it **also** depends on the location, orientation, and size of aperture S.

For **example**, if the aperture size approaches **zero**, the power P flowing through the aperture **likewise** approaches zero. This of course does **not** mean that the power radiated by the antennas is zero—it is **likely very large**.

However, there are surfaces S were the surface integration **does** tell us precisely the **radiated power**!

Consider now a **closed** surface—one that **completely surrounds** the radiating antenna.

It turns out that integrating the power density across this **closed** surface **will** tell us the power radiated by the antenna:

$$\boxed{P_{rad} = \oint\!\!\oint_S \mathbf{W}(\bar{r}) \cdot \overline{ds}}$$

Q: *Yikes,* ***why*** *does this work? What's so special about a* ***closed*** *surface?*

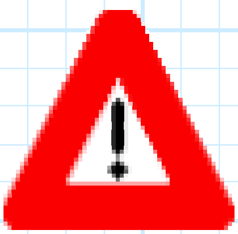
A: Essentially, this works because of **conservation of energy**.

Remember, an antenna propagates electromagnetic energy **outward** from the antenna. This energy flows in **all possible directions** (although not uniformly in all directions!).

If we integrate the power density across a surface that **completely surrounds** the antenna, then we are "capturing" the energy flowing outward in **all** possible directions.

→ There are **no "holes"** in a **closed** surface!

Our answer thus describes the **total power** flowing outward from the antenna. By conservation of energy, this **must** be equal to the power being **radiated by the antenna**!



There is **one** important caveat to this statement. The volume surrounded by closed surface S must be **lossless**. If there is lossy material in this volume, then some of the radiated power will be **absorbed** by this material, and thus will **not** propagate through closed surface S. In this case we will find:

$$P_{rad} > \oiint_S \mathbf{W}(\bar{r}) \cdot \overline{ds} \qquad \textbf{if} \text{ volume is lossy}$$

But, we will **assume** that the volume is **not** lossy, that it is essentially **free-space** (e.g., a clear atmosphere).

Q: *But what should the closed surface S* ***be****? Doesn't the integration* ***depend*** *on the size/shape of closed surface S?*

A: Although this would **seem** to be the case, it is in fact **not**. **Any** and **all** closed surfaces that surround the antenna will in fact provide the **same answer** for the surface integration.

→ After all, there is only **one** correct value of P_{rad}!

Q: *Since it doesn't matter, can I just choose **any** closed surface S that surrounds the antenna.*

A: Theoretically yes, but being efficient (i.e., **lazy**) engineers, we might choose a surface S that makes the surface integration as **easy as possible**.

This closed surface is simply a **sphere**, centered at the **origin** (i.e., centered at the antenna location). To see how this simplifies things, consider a spherical surface S, with radius a.

This **surface** is thus described as:

$$r = a \qquad 0 \leq \theta \leq \pi \qquad 0 \leq \phi \leq 2\pi$$

and $\overline{ds} = \hat{\mathbf{r}}\ r^2 d\theta\, d\phi$.

Given that the radiated **power density** has the form:

$$\mathbf{W}(\bar{r}) = U(\theta,\phi)\frac{\hat{\mathbf{r}}}{r^2}$$

we find that the **surface integral** is:

$$\oint\!\!\oint_S \mathbf{W}(\bar{r}) \cdot \overline{ds} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi) \frac{1}{r^2}\, \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\; r^2 d\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\, d\theta d\phi$$

Thus, we find that we can **always** determine the radiated power by integrating over the **radiation intensity** function produced by the antenna:

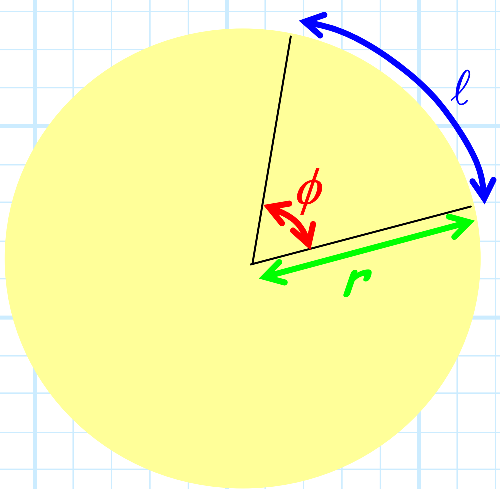
$$\boxed{P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\, d\theta d\phi}$$

The Steradian

Q: *What the heck is a steradian?*

A: First, let's examine what a **radian** is!

Recall for a **circle** (a two-dimensional object), we find:



$$\ell = \phi\, r$$

where:

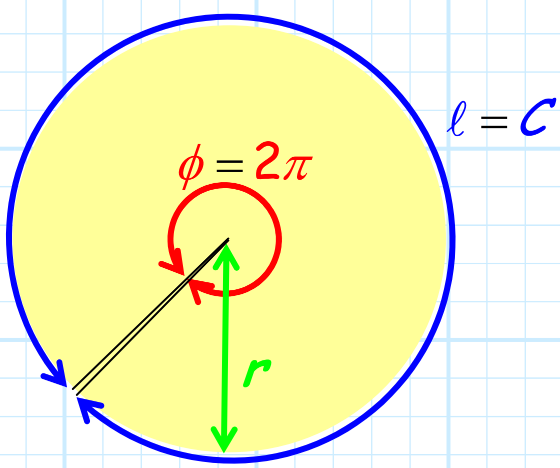
$\ell =$ arc length $[m]$

$\phi =$ angle $[radians]$

$r =$ radius $[m]$

An arc length that entirely surrounds the circle would be the circle's **circumference** (C). The angle ϕ that subtends this arc length is of course be 2π, and inserting these values into the equation above gives:

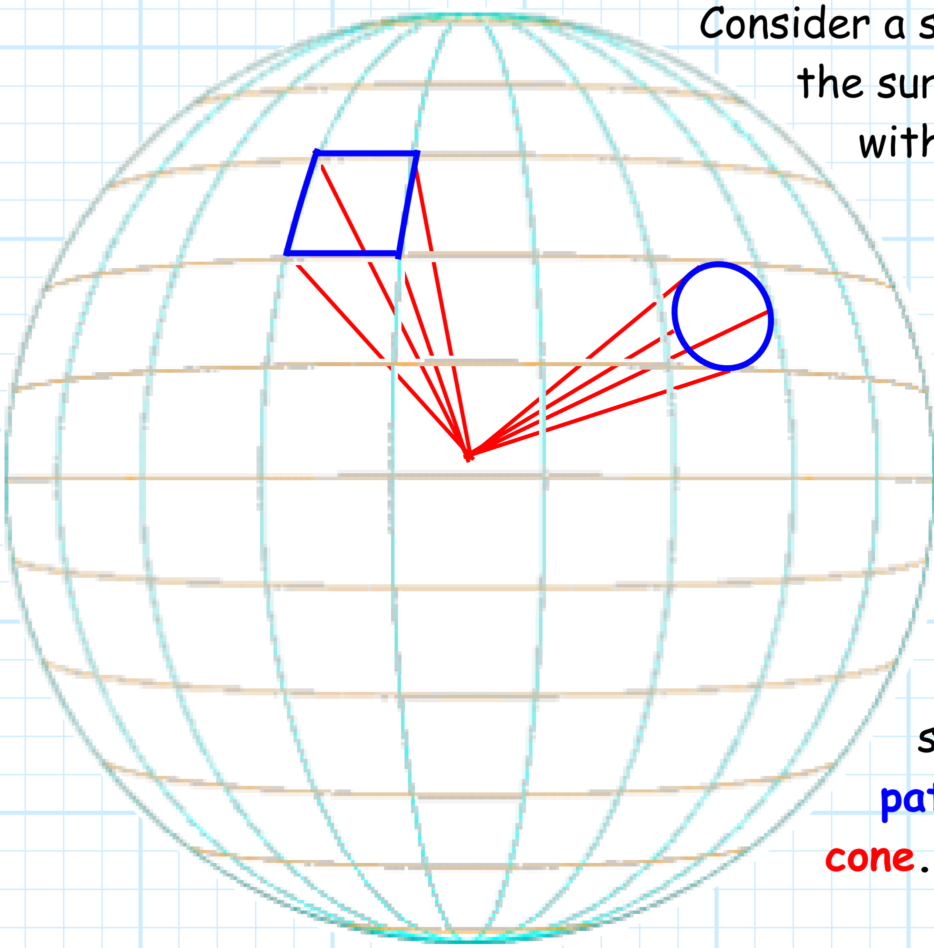
$$C = 2\pi\, r$$



Look **familiar**?

Q: *But wait! A circle is a* ***two****-dimensional structure, yet we (along with our* ***antennas****) live in a* ***three****-dimensional world.*

A: True enough! The **3**-dimensional equivalent of a circle is a **sphere**.



Consider a small **patch** on the surface of the sphere, with surface area *A* .

Each patch is subtended by a **cone**, whose point begins at the **center** of the sphere.

The larger the surface area of the **patch**, the larger the **cone**.

We say this cone forms a **solid angle** Ω, and the size of this cone is expressed in **Steradians**!

Q: *How do we determine the **size** of a solid angle in **steradians**?*

A: The size Ω of a **solid angle** that subtends a section of surface area on a sphere with radius r is:

$$\Omega = \frac{A}{r^2}$$

where A the **area** of the subtended surface (i.e., the area of the "patch") and Ω is expressed in **steradians**.

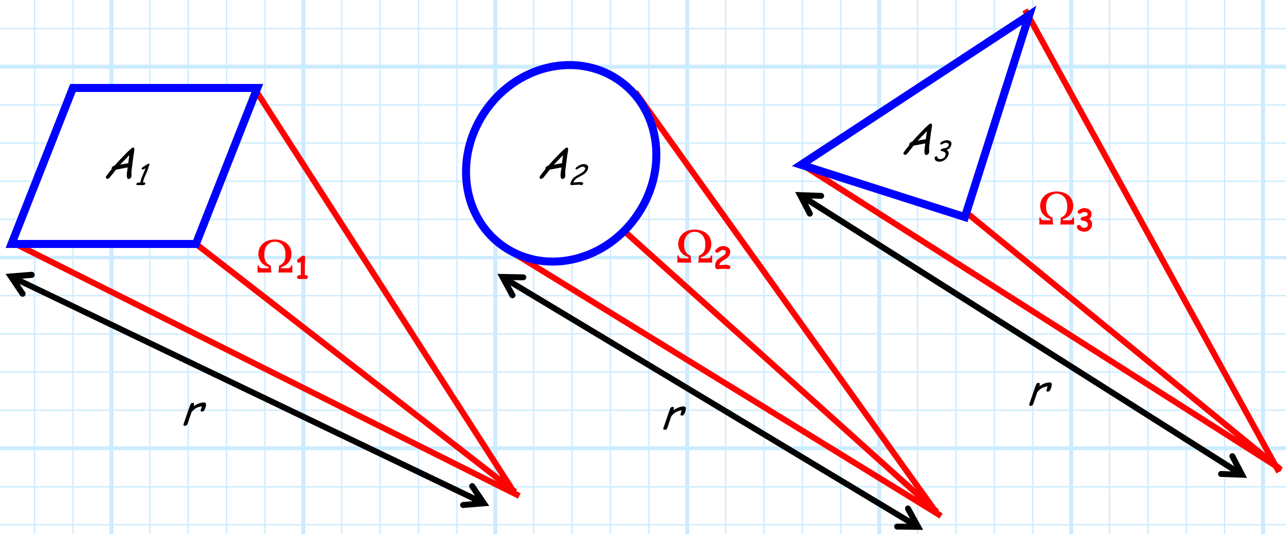
Be careful! The **units** of A and r **must** be the same! For example if A = 100 $\mathbf{m^2}$ then r must be expressed in **meters**. If r =5 **kilometers** then A must be expressed in $\mathbf{km^2}$.

Now, we can rewrite the above equation into its more **common** form:

$$\boxed{A = \Omega\, r^2}$$

Note that neither the **shape** of the subtended patch of surface, nor the **shape** of the resulting solid angle, matters in the above relationship.

In other words the subtended patch could be a **circle**, **triangle**, **square**, or any other shape. The result above would be the **same**!



Therefore, if $A_1 = A_2 = A_3$, then $\Omega_1 = \Omega_2 = \Omega_3$. Even though they have **different shapes**, they have precisely the **same size**!

Q: *What if the solid angle gets so large that it subtends the* ***entire surface*** *of the sphere? What is the* ***size*** *of the solid angle then???*

A: Recall the surface of an **entire** sphere has **area**:

$$A = 4\pi r^2$$

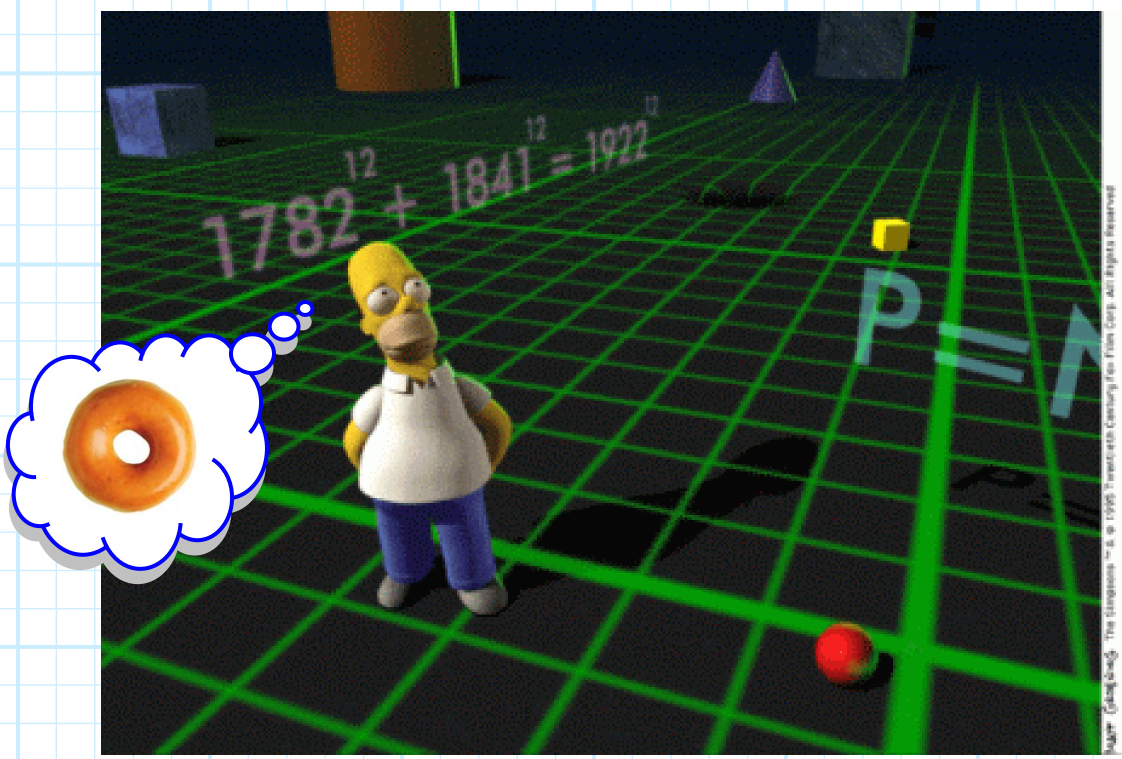
We likewise know that:

$$A = \Omega r^2$$

Equating these two, we find that a **solid angle** that subtends the **entire** surface of a sphere must have a size of $\mathbf{4}\pi$ **steradians**!

Thus, we conclude that a **planar angle** of **2π radians** subtends the entire **circumference** of a **circle**, but a **solid angle** of **4π steradians** subtends the entire **surface** of a **sphere**.

Note this means that a solid angle of **2π steradians** subtends the surface of a **hemisphere.**



Radiation Intensity

We found that the **power density** of a spherical wave produced by an **antenna** located at the origin has the form:

$$\mathbf{W}(r,\theta,\phi) = U(\theta,\phi)\frac{\hat{\mathbf{r}}}{r^2}$$

and that the **total radiated power** from an antenna located at the origin is:

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\, sin\,\theta\; d\theta\, d\phi$$

Q: *This "**radiation intensity**" $U(\theta,\phi)$ seems to be **very** important. What does it **indicate**? Does it have any physical **meaning**?*

A: It turns out that an antenna does **not** (in fact, it **cannot**) radiate power **uniformly** in all directions. Rather, an antenna distributes power **unequally**—more power in some directions and less power in others.

The radiation intensity $U(\theta,\phi)$ **describes** this unequal **distribution of power**—it tells how the radiated power is distributed as a function of radiation **direction**.

Q: *What are the **units** of the radiation intensity function?*

A: Radiation intensity is expressed in units of—
Watts/**steradian**!

To see why, consider the (impossible) case where an antenna **does** distribute radiated power **equally** in all directions. We call this **isotropic radiation**.

The intensity in this case is thus a **constant**:

$$U(\theta,\phi) = U_0$$

Although this isotropic intensity function is physically **impossible**, it will help illustrate the **physical meaning** of intensity.

We find that the **radiated power** for this case is:

$$\begin{aligned} P_{rad} &= \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\, sin\theta\, d\theta\, d\phi \\ &= \int_0^{2\pi}\int_0^{\pi} U_0\, sin\theta\, d\theta\, d\phi \\ &= U_0 \int_0^{2\pi}\int_0^{\pi} sin\theta\, d\theta\, d\phi \\ &= U_0\,(2\pi)(2) \\ &= 4\pi\, U_0 \end{aligned}$$

We can rearrange this result to determine that the **intensity** produced by an **isotropic radiator** is:

For isotropic radiator **only**!

$$U(\theta,\phi) = U_0 = \frac{P_{rad}}{4\pi} \qquad \left[\frac{Watts}{steradian}\right]$$

Q: *What's up with that 4π in the denominator? Does it have any significance?*

A: Absolutely! Look again at the units of intensity—**Watts**/**Steradian**. **Compare** this to the expression for **isotropic** radiation intensity:

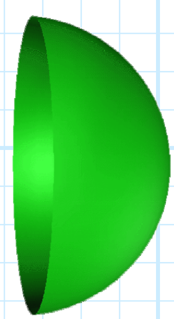
$$U_0 = \frac{P_{rad}}{4\pi} \qquad \left[\frac{Watts}{steradian}\right]$$

In other words, the antenna radiates P_{rad} Watts **uniformly** throughout a **solid angle** of 4π steradians.

Q: *4π steradians! Isn't that the size of a solid angle that subtends an **entire** sphere?*

A: You bet! 4π steradians is the **largest possible** solid angle, one that includes **all** possible directions θ and ϕ.

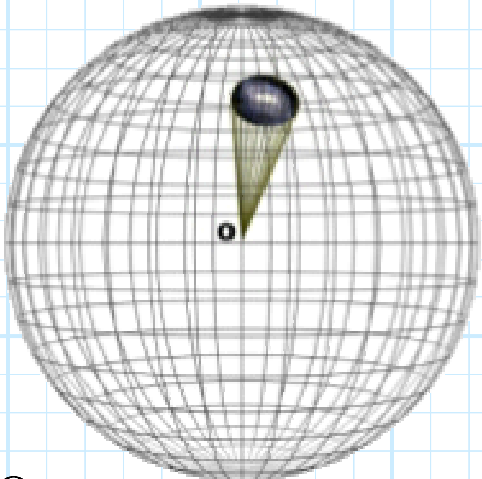
Now, say that an antenna radiates **all** its power uniformly throughout a one **hemisphere** (and thus **no power** into the **other** hemisphere).



The radiation intensity in one hemisphere (with a solid angle of 2π steradians) is therefore $P_{rad}/2\pi$, and zero in the other:

$$D(\theta,\phi) = \begin{cases} \dfrac{P_{rad}}{2\pi} & \text{in one hemisphere} \\ 0 & \text{in the other hemisphere} \end{cases} \qquad \left[{}^{W}\!/_{strd}\right]$$

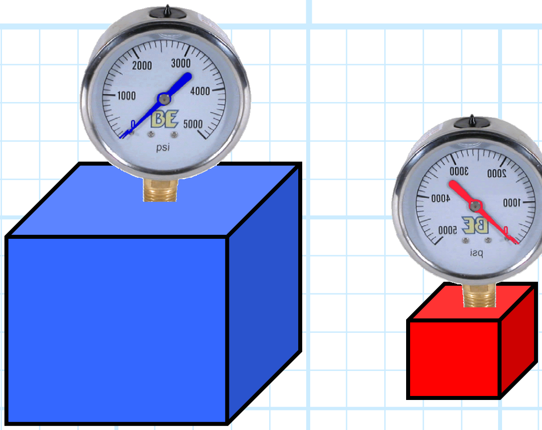
Note that the antenna in this case places **all** its radiated power in a solid angle **half** the size of the isotropic case. As a result, the **intensity** in the solid angle is **twice** as large as the isotropic case!



Or, if an antenna radiates **all** its power **uniformly** throughout a solid angle of Ω steradians, the radiation intensity **within** this solid angle will be P_{rad}/Ω, and zero **outside** the solid angle:

$$D(\theta,\phi) = \begin{cases} \dfrac{P_{rad}}{\Omega} & \text{inside solid angle } \Omega \\ 0 & \text{outside solid angle } \Omega \end{cases} \qquad \left[{}^{W}\!/_{strd}\right]$$

Note that as the solid angle gets **smaller**, the intensity will **increase** (assuming P_{rad} remains unchanged). As the antenna "focuses" its power into a smaller and smaller **cone**, the **intensity** in that cone will get larger and larger.



This is somewhat(?) analogous to **compressing** a fixed amount of **gas** into a smaller and smaller **volume**—the **pressure** within the volume will get more **intense**!

These are **simple examples** to illustrate the **meaning** of radiation intensity $U(\theta,\phi)$. However, we find that the radiation intensity of **real** antennas will be **continuous** functions of θ and ϕ. For **example**:

$$D(\theta,\phi) = 10.0 \cos^2\theta \sin\phi$$

Q: *I'm a bit* ***confused****. What's the* ***difference*** *between radiation intensity $U(\theta,\phi)$ and power density $\mathbf{W}(\bar{r})$?*

A: Recall the two are **related** by the expression:

$$\mathbf{W}(r,\theta,\phi) = U(\theta,\phi)\frac{\hat{\mathbf{r}}}{r^2}$$

From this expression we note these **differences**:

1. Radiation Intensity $U(\theta,\phi)$ is a **scalar** quantity, while power density $\mathbf{W}(\bar{r})$ is a **vector** quantity.

2. Radiation Intensity $U(\theta,\phi)$ is a function of coordinates θ and ϕ **only**, while power density $\mathbf{W}(\bar{r})$ is a function of **all three** spherical coordinates r,θ,ϕ.

From these observations we can conclude:

Radiation Intensity $U(\theta,\phi)$ is a description of how an **antenna** (located at $r=0$) **behaves**—how it distributes energy across different directions defined by coordinates θ and ϕ.

Power density $\mathbf{W}(\bar{r})$ is a description of the propagating (spherical) **electromagnetic wave** created by the antenna. It is defined at **all points** in the universe—we can determine the magnitude **and** direction of the power density at any location in space—a location denoted by coordinates r,θ,ϕ.

Finally, let's again consider the **mythical** isotropic radiator. We know that the **intensity** of such a radiator will be **uniform** across all directions:

$$U(\theta,\phi) = U_0 = \frac{P_{rad}}{4\pi} \qquad \left[\frac{Watts}{steradian}\right]$$

And thus the **power density** created by this isotropic radiator will be:

$$\begin{aligned}\mathbf{W}(r,\theta,\phi) &= U(\theta,\phi)\frac{\hat{\mathbf{r}}}{r^2} \\ &= U_0\frac{\hat{\mathbf{r}}}{r^2} \\ &= \frac{P_{rad}}{4\pi}\frac{\hat{\mathbf{r}}}{r^2} \\ &= \frac{P_{rad}}{4\pi r^2}\hat{\mathbf{r}}\end{aligned}$$

For isotropic radiator **only**!

Note that this makes **perfect sense**! 

Consider a **sphere**, centered at the origin, with **radius** r. At the center of this sphere is an isotropic radiator, and thus the **power density must be precisely the same at every point on the surface of this sphere**.

Q: *Why is that?*

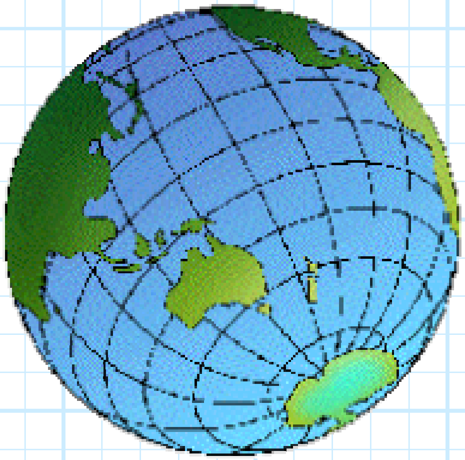
A: Remember, the radiation of an isotropic radiator is independent of θ and ϕ (i.e., the **same in all directions**), and **every** point on the sphere is precisely the **same distance** from the radiator (the distance r). The power density thus must likewise be a **constant** across this entire spherical **surface**.

Q: *But what is the **value** of this constant?*

A: We simply take the **power** flowing through this sphere (i.e., P_{rad} by conservation of energy), and **divide** it by the **surface area** of the sphere. Recall the surface area of this sphere is $4\pi r^2$!

$$\mathbf{W}(r,\theta,\phi) = \frac{P_{rad}}{4\pi r^2}\,\hat{\mathbf{r}} \qquad \frac{\textit{Watts}}{\textit{unit area}}$$

Spherical Coordinates

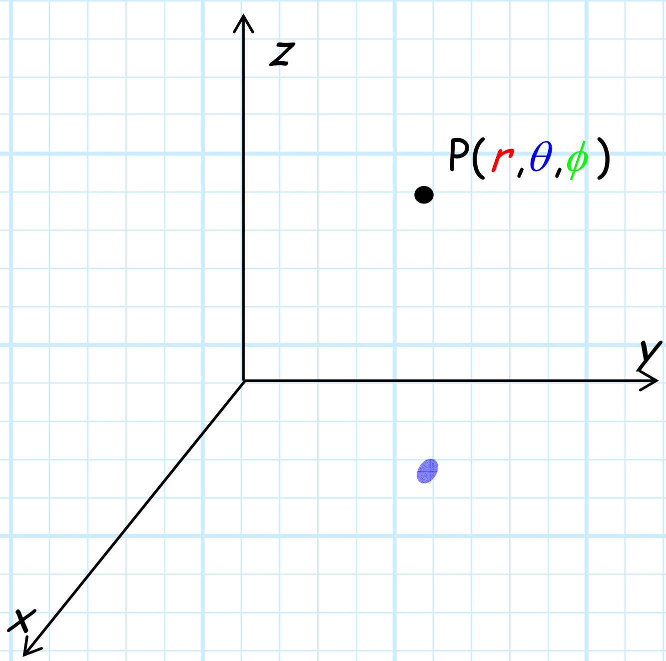


* **Geographers** specify a location on the Earth's surface using **three** scalar values: **longitude**, **latitude**, and **altitude**.

* Both longitude and latitude are **angular** measures, while altitude is a measure of **distance**.

* Latitude, longitude, and altitude are similar to **spherical coordinates**.

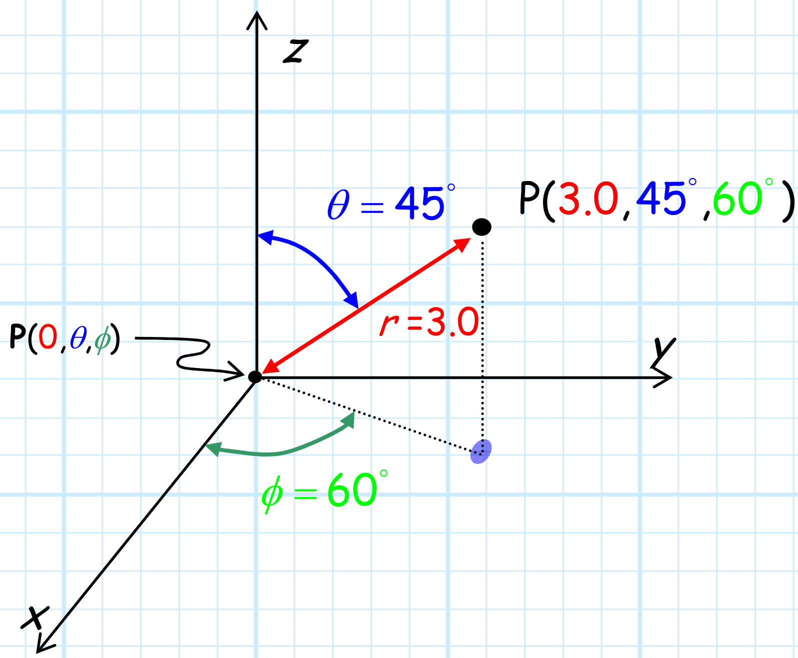
* Spherical coordinates consist of one scalar value (r), with units of **distance**, while the other two scalar values (θ,ϕ) have **angular** units (degrees or radians).



1. For spherical coordinates, r $(0 \le r < \infty)$ expresses the **distance** of the point from the **origin** (i.e., similar to **altitude**).

2. Angle θ $(0 \le \theta \le \pi)$ represents the angle formed **with the *z*-axis** (i.e., similar to **latitude**).

3. Angle ϕ $(0 \le \phi < 2\pi)$ represents the rotation angle around the z-axis, **precisely** the same as the **cylindrical** coordinate ϕ (i.e., similar to **longitude**).



Thus, using **spherical** coordinates, a point in space can be unambiguously defined by **one distance** and **two angles**.