Variance-Time Curve for Packet Streams Generated by Exponentially Distributed ON/OFF Sources

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Abstract—**In this letter we provide a solution to an open problem in network traffic characterization. Specifically we present a closed-form expression of the variance-time curve for a packet stream generated by exponentially distributed ON/OFF sources. So far, the variance-time curve for such processes was obtained by numerical analysis at the desired time scales. We also show that under low and medium loads, the variance-time curve obtained by approximating the ON/OFF/Exponential source as a stationary fluid source is over-estimated. Lastly, as a by-product of our analysis, we present a new mathematical identity based on the incomplete Gamma function.**

Index Terms—**Traffic characterization, variance-time curve, ON/OFF/exponential source.**

I. Introduction

TRAFFIC is the driving force behind all telecommunication activities, and models are of crucial importance for evaluating network performance. In this letter we consider a traditional model, namely the ON/OFF Exponential traffic source model. This traffic model became popular when it was used in [1] to characterize the aggregate packet arrival process generated by the superposition of separate voice streams. For computing the index of dispersion for counts (IDC) for an aggregated voice-packet stream, the variance-time curve was obtained in [2] by numerical analysis at the desired time scales.

Our main goal in this letter is to derive a closed-form expression of the variance-time curve for the packet streams generated by such traffic sources. This closed-form expression of the variance-time is very useful for better characterizing the variability of packet streams that are generated by such traffic sources, i.e., VoIP packet traffic sources. This exact expression of the variance-time curve is then compared with an approximated expression derived when considering the ON/OFF/Exponential sources as a stationary fluid source.

II. ON/OFF Exponential Source Model

We consider a packet stream (Fig. 1) generated by a single ON/OFF traffic source with strictly alternating ON- and OFF-periods. The OFF-periods are exponentially distributed with mean $\frac{1}{\beta}$, and during the ON-periods, the source transmits a packetized message whose size is also exponentially distributed. Hence, the number of packets (W) transmitted during ON times is geometrically distributed, and thus the packet stream due to a single source is a renewal process. Assuming that the size of all packets transmitted is fixed, the probability density function (pdf) for the packet interarrival times is given by [1]

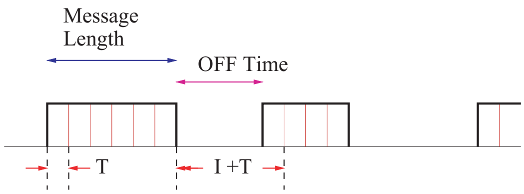

Fig. 1. Packet stream generated by a single ON/OFF source.

$$f(x) = p\delta(x-T) + (1-p)\beta e^{-\beta(x-T)}u(x-T) \quad (1)$$

where $u(t)$ is the unit step function, T the packet transmission time, and p $(0 < p < 1)$ the probability that the next interarrival time[1] is T. The interarrival time is measured from the last arrival bit of the $(n-1)^{st}$ packet to the last arrival bit of the n^{th} packet. Define $E[W]$ as the mean number of packets transmitted during the ON periods, then

$$p = \frac{E[W]-1}{E[W]}.$$

That is, the interarrival times $X_1, X_2, \ldots, X_n$ are of length T with probability p and of length $I+T$ with probability $1-p$, where I is the exponentially distributed random length of the OFF periods. The mean packet arrival rate is therefore

$$\lambda = \frac{1}{E[X]} = \frac{1}{\int_0^\infty xf(x)\,dx} = \frac{\beta}{(1-p)+\beta T}. \quad (2)$$

Let α^{-1} to be the mean ON time. Then $\alpha^{-1} = E[W]T$, and since $E[W] = \frac{1}{1-p}$, we have that $\alpha T = 1-p$.

III. Deriving the Variance-Time Curve

Assuming an arbitrary origin, the renewal process $\{N(t), t \geq 0\}$ is stationary [3], where $N(t)$ denotes the number of packet arrivals in an interval of length t. The variance of arrival counts over a time interval of length t, or variance-time curve, is

$$\begin{aligned} Var[N(t)] &= E[N^2(t)] - (E[N(t)])^2 \\ &= E[N(t)\{N(t)+1\}] - E[N(t)] - (E[N(t)])^2 \\ &= \Psi(t) - \lambda t - (\lambda t)^2 \end{aligned} \quad (3)$$

where the challenge here is to obtain

$$\Psi(t) = E[N(t)\{N(t)+1\}]. \quad (4)$$

[1]We assume that packets from the same message are transmitted back to back without any inter-idle time.

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It is known from [4] that on taking the Laplace transform of $\Psi(t)$ we get

$$\Psi^*(s) = \mathcal{L}[\Psi(t)] = \frac{2\lambda}{s^2\{1 - f^*(s)\}} \qquad for\ Re[s] > 0, \quad (5)$$

where

$$\begin{aligned} f^*(s) &= \mathcal{L}[f(x)] = \int_0^\infty f(x)e^{-sx}\,dx \\ &= \left[p + (1-p)\frac{\beta}{s+\beta}\right] e^{-sT} \end{aligned} \quad (6)$$

is the Laplace transform of $f(x)$. Hence,

$$\Psi^*(s) = \frac{2\lambda}{s^2\left\{1 - \left[p + (1-p)\frac{\beta}{s+\beta}\right]e^{-sT}\right\}} = \frac{\varphi^*(s)}{s^2} \quad (7)$$

by letting

$$\varphi^*(s) = \frac{2\lambda}{1 - \left[p + (1-p)\frac{\beta}{s+\beta}\right]e^{-sT}}. \quad (8)$$

The next step in getting $\Psi(t)$ is to compute the inverse Laplace transform of $\Psi^*(s)$

$$\Psi(t) = \mathcal{L}^{-1}[\Psi^*(s)] = \int_{c-j\infty}^{c+j\infty} \Psi^*(s)e^{st}\,ds \quad (9)$$

where $Re[s] = c > 0$ being chosen so that all singularities of $\Psi^*(s)$ lie to the left of the line of integration.

Now, let $s = a + jb$ where $a = Re[s]$ and $b = Im[s]$. Then

$$\begin{aligned} |f^*(s)| &= \left|\left[p + (1-p)\frac{\beta}{s+\beta}\right]e^{-sT}\right| \\ &= \left|\left[p + (1-p)\frac{\beta}{a+jb+\beta}\right]e^{-(a+jb)T}\right| \\ &= \left|p + (1-p)\frac{\beta}{a+\beta+jb}\right|\left|e^{-(a+jb)T}\right| \\ &= \left|\frac{pa + p\beta + jpb + \beta - p\beta}{a+\beta+jb}\right|e^{-aT} \\ &= \left|\frac{pa + \beta + jpb}{a+\beta+jb}\right|e^{-aT}. \end{aligned}$$

Clearly, $e^{-aT} < 1 \ \ \forall\, a = Re[s] > 0$ and

$$\left|\frac{pa+\beta+jpb}{a+\beta+jb}\right| = \sqrt{\frac{(pa+\beta)^2 + (pb)^2}{(a+\beta)^2 + b^2}} < 1$$

for

$$\begin{aligned} &\forall\, 0 < p < 1,\ \beta > 0, \\ &\forall\, a = Re[s] > 0,\ and \\ &\forall\, b = Im[s], \end{aligned}$$

making

$$|f^*(s)| < 1 \qquad for\ Re[s] > 0. \quad (10)$$

Thus, for $Re[s] > 0$

$$\varphi^*(s) = \frac{2\lambda}{1 - f^*(s)}$$

is a geometric series[2]. Therefore, on the line of integration $Re[s] = c > 0$ in (9) $|f^*(s)| < 1$ and the function $\varphi^*(s)$ is equal to

$$\sum_{n=0}^{\infty} 2\lambda\,[f^*(s)]^n = 2\lambda \sum_{n=0}^{\infty}\left[p + (1-p)\frac{\beta}{s+\beta}\right]^n e^{-snT}. \quad (11)$$

Upon using the Binomial Theorem[3] in (11) we get

$$\begin{aligned} \varphi^*(s) &= 2\lambda\sum_{n=0}^{\infty}\sum_{z=0}^{n}\binom{n}{z}p^{n-z}\left[(1-p)\frac{\beta}{s+\beta}\right]^z e^{-snT} \\ &= 2\lambda\sum_{n=0}^{\infty}p^n e^{-snT} + \\ &\quad 2\lambda\sum_{n=1}^{\infty}\sum_{z=1}^{n}\binom{n}{z}p^{n-z}(1-p)^z\beta^z\frac{e^{-snT}}{(s+\beta)^z}. \end{aligned} \quad (12)$$

Letting

$$\varphi_1^*(s) = 2\lambda\sum_{n=0}^{\infty}p^n e^{-snT}$$

and

$$\varphi_2^*(s) = 2\lambda\sum_{n=1}^{\infty}\sum_{z=1}^{n}\binom{n}{z}p^{n-z}(1-p)^z\beta^z\frac{e^{-snT}}{(s+\beta)^z},$$

then $\varphi^*(s) = \varphi_1^*(s) + \varphi_2^*(s)$ and

$$\Psi^*(s) = \frac{\varphi^*(s)}{s^2} = \frac{\varphi_1^*(s)}{s^2} + \frac{\varphi_2^*(s)}{s^2} = \Psi_1^*(s) + \Psi_2^*(s).$$

Obviously, $\Psi(t) = \Psi_1(t) + \Psi_2(t)$. Using the Laplace transform pair [5]

$$\frac{e^{-as}}{s^2} \overset{\mathcal{L}}{\Longleftrightarrow} (t-a)u(t-a) \qquad a \geq 0$$

we easily obtain

$$\Psi_1(t) = 2\lambda\sum_{n=0}^{\infty}p^n(t-nT)u(t-nT). \quad (13)$$

Similarly, by using the Laplace transform pair

$$\begin{aligned} \frac{e^{-snT}}{s^2(s+a)^z} \overset{\mathcal{L}}{\Longleftrightarrow} &\ \frac{1}{a^{z+1}}\{a(t-nT)G[a(t-nT), z] - \\ &\ zG[a(t-nT), z+1]\}\,u(t-nT) \end{aligned}$$

for $a, nT \geq 0$ and $z = 1, 2, \ldots$, where

$$G(x,y) = \frac{1}{\Gamma(y)}\int_0^x t^{y-1}e^{-t}\,dt \qquad y > 0, \quad x > 0$$

is the incomplete Gamma[4] function, we get

$$\begin{aligned} \Psi_2(t) &= 2\lambda\sum_{n=1}^{\infty}\sum_{z=1}^{n}\binom{n}{z}\frac{p^{n-z}(1-p)^z}{\beta}\{\beta(t-nT)\,\times \\ &\quad G[\beta(t-nT), z] - zG[\beta(t-nT), z+1]\}u(t-nT). \end{aligned} \quad (14)$$

A slightly different form of $\Psi_2(t)$ is presented in the Appendix, as well as a new mathematical identity based on the incomplete Gamma function derived by comparing the two forms of $\Psi_2(t)$.

[2]Geometric Series: $\sum_{n=0}^{\infty}\alpha x^n = \frac{\alpha}{1-x} \quad for\ |x| < 1$.

[3]Binomial Theorem: $(a+x)^n = \sum_{z=0}^{n}\binom{n}{z}a^{n-z}x^z = a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}x^2 + \cdots + x^n$.

[4]Gamma Function: $\Gamma(y) = \int_0^\infty t^{y-1}e^{-t}\,dt \qquad y > 0$.

Putting everything together, we then obtain the variance-time curve as:

$$\begin{aligned} Var[N(t)] &= \Psi_1(t) + \Psi_2(t) - \lambda t - (\lambda t)^2 \\ &= 2\lambda \sum_{n=0}^{\infty} p^n (t - nT) u(t - nT) + 2\lambda \sum_{n=1}^{\infty} \sum_{z=1}^{n} \binom{n}{z} \\ &\times \frac{p^{n-z}(1-p)^z}{\beta} \{\beta(t - nT) G\left[\beta(t - nT), z\right] \\ &- zG\left[\beta(t-nT), z+1\right]\} u(t - nT) - \lambda t - (\lambda t)^2. \end{aligned} \tag{15}$$

Suppose now an aggregated packet stream is generated by the superposition of M independent but not necessarily identical ON/OFF/Exponential sources. Assuming again stationarity, then the index of dispersion for counts is given by

$$IDC(t) = \frac{\sum_{i=1}^{M} Var[N_i(t)]}{\sum_{i=1}^{M} \lambda_i t}$$

where λ_i is the mean packet arrival rate and $N_i(t)$ is the number of packet arrivals in the interval $(0, t]$ from the i^{th} source. In case that the sources are also identical, then the IDC for the aggregated packet stream is identical to that of a single source.

IV. Comparison with the Fluid Source Model

Considering the ON/OFF/Exponential traffic source as a stationary fluid source with a constant transmission rate $\nu = \frac{1}{T}$ during the ON-periods, it can be described by a two-state Markov process. Letting $\rho = \alpha + \beta$, we then have from [6], [7] the approximate[5] variance-time curve as

$$\tilde{V}ar[N(t)] = \frac{2(1-p)\lambda^3}{\beta^2} \left[t - \frac{1}{\rho}\left(1 - e^{-\rho t}\right)\right]. \tag{16}$$

One way to see that $\tilde{V}ar[N(t)]$ is an approximate of $Var[N(t)]$ given by (15) is by checking whether in the limit the index of dispersion for counts is equal with the squared coefficient of variation of the interarrival times[6], $\mathcal{C}^2(X)$: $\lim_{t\to\infty} I\tilde{D}C(t) = \frac{2}{1+p}\mathcal{C}^2(X)$. From this, the following is easily obtained:

$$\lim_{t\to\infty} \frac{\tilde{V}ar[N(t)]}{Var[N(t)]} = \frac{2}{1+p}. \tag{17}$$

Clearly, as the mean number of packets transmitted during the ON periods ($E[W]$) increases, and thus $p \to 1$, $\tilde{V}ar[N(t)] \to Var[N(t)]$ for large enough t. But as $E[W]$ decreases and thus $p \to 0$, $\tilde{V}ar[N(t)] \to 2Var[N(t)]$ for large enough t. This shows that under low and medium loads, network performance obtained by using fluid analysis is over-estimated.

[5] During the ON-periods the burst of consecutive packets generated by the source is considered as a continuous fluid. Denote $I(t) = 1_{\{source\ is\ ON\ at\ time\ t\}}$, then the cumulative number of packets generated by the source in time interval $(0, t]$ is approximately $\tilde{N}(t) = \int_0^t \nu I(t)\, ds$.

[6] Since $N(t)$ is a renewal process, then $\lim_{t\to\infty} IDC(t) = \lim_{t\to\infty}(Var[N(t)]/E[N(t)]) = \mathcal{C}^2(X) = Var[X]/(E[X])^2 = \lambda^2(1-p^2)/\beta^2$.

V. Conclusion

We derived an exact expression of the variance-time curve using point processes analysis for packet streams generated by exponentially distributed ON/OFF network traffic sources. In addition, we showed that the fluid analysis over-estimates the variance-time curve under low or medium load conditions. Finally, our analysis generated a new mathematical identity based on the incomplete Gamma function.

Appendix

Let $\xi_2^*(s) = \frac{\varphi_2^*(s)}{s}$, so that $\Psi_2^*(s) = \frac{\varphi_2^*(s)}{s^2} = \frac{\xi_2^*(s)}{s}$ and

$$\xi_2^*(s) = 2\lambda \sum_{n=1}^{\infty} \sum_{z=1}^{n} \binom{n}{z} p^{n-z}(1-p)^z \beta^z \frac{e^{-snT}}{s(s+\beta)^z}.$$

Applying the following Laplace transform pair [5]

$$\frac{e^{-snT}}{s(s+a)^z} \overset{\mathcal{L}}{\Longleftrightarrow} \frac{1}{a^z}\left\{1 - e^{-a(t-nT)} \sum_{m=0}^{z-1} \frac{[a(t-nT)]^m}{m!}\right\} u(t-nT)$$

for $a, nT \geq 0$ and $z = 1, 2, \ldots$, we get

$$\begin{aligned} \xi(t) &= 2\lambda \sum_{n=1}^{\infty} \sum_{z=1}^{n} \binom{n}{z} p^{n-z}(1-p)^z \times \\ &\left\{1 - e^{-\beta(t-nT)} \sum_{m=0}^{z-1} \frac{[\beta(t-nT)]^m}{m!}\right\} u(t-nT). \end{aligned}$$

From this, an alternative form of $\Psi_2(t)$ is obtained as follows,

$$\begin{aligned} \Psi_2(t) &= \int_0^t \xi(x)\, dx = 2\lambda \sum_{n=1}^{\infty} \sum_{z=1}^{n} \binom{n}{z} \frac{p^{n-z}(1-p)^z}{\beta} \times \\ &\left\{\beta(t-nT) - \sum_{m=0}^{z-1} G[\beta(t-nT), m]\right\} u(t-nT). \end{aligned} \tag{18}$$

Comparing the two different forms of $\Psi_2(t)$ shown in (15) and (18) we easily obtain the following identity

$$\sum_{m=1}^{z} G(x, m) = x\left[1 - G(x, z)\right] + zG(x, z+1) \qquad z = 1, 2, 3, \ldots. \tag{19}$$

To the best of authors' knowledge the above identity has never been published before.

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