7-3 The Biot-Savart Law and the Magnetic Vector Potential

Reading Assignment: *pp. 208-218*

Q: Given some **field** $\mathbf{B}(\bar{r})$, how can we determine the **source** $\mathbf{J}(\bar{r})$ that created it?

A: Easy! → $\mathbf{J}(\bar{r}) = \nabla \times \mathbf{B}(\bar{r}) / \mu_0$

Q: OK, given some **source** $\mathbf{J}(\bar{r})$, how can we determine what **field** $\mathbf{B}(\bar{r})$ it creates?

A:

HO: The Magnetic Vector Potential

HO: Solutions to Ampere's Law

HO: The Biot-Savart Law

Example: The Uniform, Infinite Line of Current

HO: B-field from an Infinite Current Sheet

The Magnetic Vector Potential

From the **magnetic** form of Gauss's Law $\nabla \cdot \mathbf{B}(\bar{r}) = 0$, it is evident that the **magnetic flux density** $\mathbf{B}(\bar{r})$ is a **solenoidal** vector field.

Recall that a solenoidal field is the **curl** of some other vector field, e.g.,:

$$\mathbf{B}(\bar{r}) = \nabla x \mathbf{A}(\bar{r})$$

Q: *The magnetic flux density* $\mathbf{B}(\bar{r})$ *is the curl of* ***what*** *vector field ??*

A: The **magnetic vector potential** $\mathbf{A}(\bar{r})$!

The **curl** of the magnetic vector potential $\mathbf{A}(\bar{r})$ is **equal** to the magnetic flux density $\mathbf{B}(\bar{r})$:

$$\boxed{\nabla x \mathbf{A}(\bar{r}) = \mathbf{B}(\bar{r})}$$

where:

magnetic vector potential $\doteq \mathbf{A}(\bar{r}) \quad \left[\dfrac{\text{Webers}}{\text{meter}}\right]$

Vector field $\mathbf{A}(\bar{r})$ is called the **magnetic** vector potential because of its **analogous** function to the **electric** scalar potential $V(\bar{r})$.

An **electric** field can be determined by taking the **gradient** of the **electric potential**, just as the **magnetic** flux density can be determined by taking the **curl** of the **magnetic potential**:

$$\mathbf{E}(\bar{r}) = -\nabla V(\bar{r}) \qquad\qquad \mathbf{B}(\bar{r}) = \nabla \times \mathbf{A}(\bar{r})$$

Yikes! We have a **big problem**!

There are actually (infinitely) **many** vector fields $\mathbf{A}(\bar{r})$ whose **curl** will equal an arbitrary magnetic flux density $\mathbf{B}(\bar{r})$. In other words, given some vector field $\mathbf{B}(\bar{r})$, the **solution** $\mathbf{A}(\bar{r})$ to the **differential equation** $\nabla \times \mathbf{A}(\bar{r}) = \mathbf{B}(\bar{r})$ is **not unique** !

But of course, **we knew this**!

To **completely** (i.e., uniquely) specify a **vector** field, we need to specify **both** its divergence and its curl.

Well, we know the **curl** of the magnetic vector potential $\mathbf{A}(\bar{r})$ is equal to magnetic flux density $\mathbf{B}(\bar{r})$. But, what is the **divergence** of $\mathbf{A}(\bar{r})$ equal to ? I.E.,:

$$\nabla \cdot \mathbf{A}(\bar{r}) = \ ???$$

By answering this question, we are essentially **defining** $\mathbf{A}(\bar{r})$.

➡ Let's define it in so that it makes our **computations easier**!

To accomplish this, we first start by writing **Ampere's Law** in terms of magnetic vector potential:

$$\nabla x \mathbf{B}(\bar{r}) = \nabla x \nabla x \mathbf{A}(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

We recall from **section 2-6** that:

$$\nabla x \nabla x \mathbf{A}(\bar{r}) = \nabla\left(\nabla \cdot \mathbf{A}(\bar{r})\right) - \nabla^2 \mathbf{A}(\bar{r})$$

Thus, we can **simplify** this statement if we decide that the **divergence** of the magnetic vector potential is **equal to zero**:

$$\boxed{\nabla \cdot \mathbf{A}(\bar{r}) = 0}$$

We call this the **gauge equation** for magnetic vector potential. Note the magnetic vector potential $\mathbf{A}(\bar{r})$ is therefore **also** a **solenoidal** vector field.

As a result of this gauge equation, we find:

$$\nabla x \nabla x \mathbf{A}(\bar{r}) = \nabla(\nabla \cdot \mathbf{A}(\bar{r})) - \nabla^2 \mathbf{A}(\bar{r})$$
$$= -\nabla^2 \mathbf{A}(\bar{r})$$

And thus **Ampere's Law** becomes:

$$\nabla x \mathbf{B}(\bar{r}) = -\nabla^2 \mathbf{A}(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

Note the Laplacian operator ∇^2 is the **vector Laplacian**, as it operates on **vector** field $\mathbf{A}(\bar{r})$.

Summarizing, we find the magnetostatic equations in terms of **magnetic vector potential $\mathbf{A}(\bar{r})$** are:

$$\nabla x \mathbf{A}(\bar{r}) = \mathbf{B}(\bar{r})$$

$$\nabla^2 \mathbf{A}(\bar{r}) = -\mu_0 \mathbf{J}(\bar{r})$$

$$\nabla \cdot \mathbf{A}(\bar{r}) = 0$$

Note that the **magnetic** form of Gauss's equation results in the equation $\nabla \cdot \nabla x \mathbf{A}(\bar{r}) = 0$. **Why** don't we include this equation in the above list?

Compare the magnetostatic equations using the magnetic vector potential $\mathbf{A}(\bar{\mathbf{r}})$ to the electrostatic equations using the electric scalar potential $V(\bar{\mathbf{r}})$:

$$\mathbf{E}(\bar{\mathbf{r}}) = -\nabla V(\bar{r})$$

$$\nabla \cdot \mathbf{E}(\bar{\mathbf{r}}) = \frac{\rho_v(\bar{\mathbf{r}})}{\varepsilon_0}$$

Hopefully, you see that the two potentials $\mathbf{A}(\bar{\mathbf{r}})$ and $V(\bar{\mathbf{r}})$ are in many ways **analogous**.

For example, we know that we can determine a static field $\mathbf{E}(\bar{\mathbf{r}})$ created by sources $\rho_v(\bar{\mathbf{r}})$ either **directly** (from Coulomb's Law), or **indirectly** by first finding **potential** $V(\bar{\mathbf{r}})$ and then taking its derivative (i.e., $\mathbf{E}(\bar{\mathbf{r}}) = -\nabla V(\bar{r})$).

Likewise, the magnetostatic equations above say that we can determine a static field $\mathbf{B}(\bar{\mathbf{r}})$ created by sources $\mathbf{J}(\bar{\mathbf{r}})$ either **directly**, or **indirectly** by first finding **potential** $\mathbf{A}(\bar{\mathbf{r}})$ and then taking its derivative (i.e., $\nabla \times \mathbf{A}(\bar{\mathbf{r}}) = \mathbf{B}(\bar{\mathbf{r}})$).

$$\rho_v(\bar{r}) \quad \Rightarrow \quad V(\bar{r}) \quad \Rightarrow \quad \mathbf{E}(\bar{r})$$

$$\mathbf{J}(\bar{r}) \quad \Rightarrow \quad \mathbf{A}(\bar{r}) \quad \Rightarrow \quad \mathbf{B}(\bar{r})$$

Solutions to Ampere's Law

Say we **know** the **current distribution** $\mathbf{J}(\bar{r})$ occurring in some physical problem, and we wish to **find** the resulting **magnetic flux density** $\mathbf{B}(\bar{r})$.

Q: *How do we find* $\mathbf{B}(\bar{r})$ *given* $\mathbf{J}(\bar{r})$*?*

A: **Two** ways! We either **directly** solve the differential equation:

$$\nabla \times \mathbf{B}(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

Or we first solve **this** differential equation for vector field $\mathbf{A}(\bar{r})$:

$$-\nabla^2 \mathbf{A}(\bar{r}) = \mu_0 \mathbf{J}(\bar{r})$$

and **then** find $\mathbf{B}(\bar{r})$ by taking the **curl** of $\mathbf{A}(\bar{r})$ (i.e., $\nabla \times \mathbf{A}(\bar{r}) = \mathbf{B}(\bar{r})$).

It turns out that the **second** option is often the easiest!

To see why, consider the **vector Laplacian** operator if vector field $\mathbf{A}(\bar{r})$ is expressed using **Cartesian** base vectors:

$$\nabla^2 \mathbf{A}(\bar{r}) = \nabla^2 A_x(\bar{r})\,\hat{a}_x + \nabla^2 A_y(\bar{r})\,\hat{a}_y + \nabla^2 A_z(\bar{r})\,\hat{a}_z$$

We therefore write **Ampere's Law** in terms of **three** separate **scalar** differential equations:

$$\nabla^2 A_x(\bar{r}) = -\mu_0 J_x(\bar{r})$$

$$\nabla^2 A_y(\bar{r}) = -\mu_0 J_y(\bar{r})$$

$$\nabla^2 A_z(\bar{r}) = -\mu_0 J_z(\bar{r})$$

Each of these differential equations is **easily solved**. In fact, we **already know** their solution!

Recall we had the **exact** same differential equation in electrostatcs (i.e., Poisson's equation):

$$\nabla^2 V(\bar{r}) = \frac{-\rho_v(\bar{r})}{\varepsilon_0}$$

We **know** the solution $V(\bar{r})$ to **this** differential equation is:

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_V \frac{\rho_v(\bar{r}')}{|\bar{r}-\bar{r}'|}\, dv'$$

Mathematically, Poisson's equation is **exactly** the same as **each** of the three scalar differential equations at the top of the page, with these **substitutions**:

$$V(\bar{r}) \rightarrow A_x(\bar{r}) \qquad \rho_v(\bar{r}) \rightarrow J_x(\bar{r}) \qquad \frac{1}{\varepsilon_0} \rightarrow \mu_0$$

The **solutions** to the **magnetic** differential equation are therefore:

$$A_x(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{J_x(\bar{r}')}{|\bar{r} - \bar{r}'|} \, dv'$$

$$A_y(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{J_y(\bar{r}')}{|\bar{r} - \bar{r}'|} \, dv'$$

$$A_z(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{J_z(\bar{r}')}{|\bar{r} - \bar{r}'|} \, dv'$$

and since:

$$\mathbf{A}(\bar{r}) = A_x(\bar{r})\,\hat{a}_x + A_y(\bar{r})\,\hat{a}_y + A_z(\bar{r})\,\hat{a}_z$$

and:

$$\mathbf{J}(\bar{r}) = J_x(\bar{r})\,\hat{a}_x + J_y(\bar{r})\,\hat{a}_y + J_z(\bar{r})\,\hat{a}_z$$

we can **combine** these three solutions and get the **vector** solution to our **vector** differential equation:

$$\boxed{\mathbf{A}(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{r}')}{|\bar{r} - \bar{r}'|} \, dv'}$$

Therefore, given **current distribution** $\mathbf{J}(\bar{r})$, we use the above equation to determine **magnetic vector potential** $\mathbf{A}(\bar{r})$. We **then** take the **curl** of this result to determine **magnetic flux density** $\mathbf{B}(\bar{r})$.

For surface current, the resulting magnetic vector potential is:

$$\mathbf{A}(\bar{\mathbf{r}}) = \frac{\mu_0}{4\pi} \iint_S \frac{\mathbf{J}_s(\bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|} ds'$$

and for a current I flowing along contour C, we find:

$$\mathbf{A}(\bar{\mathbf{r}}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{\overline{d\ell'}}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|}$$

Again, ponder the **analogy** between these equations involving **sources** and **potentials** and the equivalent equation from **electrostatics**:

$$V(\bar{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0} \iiint_V \frac{\rho_v(\bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|} dv'$$

The Biot-Savart Law

So, we now know that given some **current density**, we can find the resulting **magnetic vector potential** $\mathbf{A}(\bar{\mathrm{r}})$:

$$\mathbf{A}(\bar{\mathrm{r}}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{\mathrm{r}}')}{|\bar{\mathrm{r}} - \bar{\mathrm{r}}'|} \, dv'$$

and then determine the resulting **magnetic flux density** $\mathbf{B}(\bar{\mathrm{r}})$ by taking the **curl**:

$$\mathbf{B}(\bar{\mathrm{r}}) = \nabla \mathrm{x} \mathbf{A}(\bar{\mathrm{r}})$$

Q: ***Golly****, can't we somehow* ***combine*** *the curl operation and the magnetic vector potential integral?*

A: Yes! The result is known as the **Biot-Savart Law.**

Combining the two above equations, we get:

$$\mathbf{B}(\bar{\mathrm{r}}) = \nabla \mathrm{x} \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{\mathrm{r}}')}{|\bar{\mathrm{r}} - \bar{\mathrm{r}}'|} \, dv'$$

This result is of course **not** very helpful, but we note that we can move the curl operation **into** the integrand:

$$\mathbf{B}(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \nabla x \frac{\mathbf{J}(\bar{r}')}{|\bar{r} - \bar{r}'|} dv'$$

Note this result **reverses** the process: **first** we perform the curl, and **then** we integrate.

We can do this is because the **integral** is over the **primed** coordinates (i.e.,$\bar{r}'$) that specify the **sources** (current density), while the **curl** take the derivatives of the **unprimed** coordinates (i.e., $\bar{r}$) that describe the **fields** (magnetic flux density).

Q: *Yikes! That curl operation still looks particularly **difficult**. How we perform it?*

A: We take advantage of a know **vector identity**! The curl of vector field $f(\bar{r})\mathbf{G}(\bar{r})$, where $f(\bar{r})$ is any **scalar** field and $\mathbf{G}(\bar{r})$ is any **vector** field, can be evaluated as:

$$\nabla x\left(f(\bar{r})\mathbf{G}(\bar{r})\right) = f(\bar{r})\nabla x\mathbf{G}(\bar{r}) - \mathbf{G}(\bar{r})x\nabla f(\bar{r})$$

Note the **integrand** of the above equation **is** in the form $\nabla x\left(f(\bar{r})\mathbf{G}(\bar{r})\right)$, where:

$$f(\bar{r}) = \frac{1}{|\bar{r} - \bar{r}'|} \qquad \text{and} \qquad \mathbf{G}(\bar{r}) = \mathbf{J}(\bar{r}')$$

Therefore we find:

$$\nabla \times \left(\frac{\mathbf{J}(\bar{r}')}{|\bar{r} - \bar{r}'|} \right) = \frac{1}{|\bar{r} - \bar{r}'|} \nabla \times \mathbf{J}(\bar{r}') - \mathbf{J}(\bar{r}') \times \nabla \left(\frac{1}{|\bar{r} - \bar{r}'|} \right)$$

In the **first** term we take the **curl** of $\mathbf{J}(\bar{r}')$. Note however that this vector field is a **constant** with respect to the **unprimed** coordinates $\bar{r}$. Thus the **derivatives** in the curl will all be equal to **zero**, and we find that:

$$\nabla \times \mathbf{J}(\bar{r}') = 0$$

Likewise, it can be shown that:

$$\nabla \left(\frac{1}{|\bar{r} - \bar{r}'|} \right) = -\frac{\bar{r} - \bar{r}'}{|\bar{r} - \bar{r}'|^3}$$

Using these results, we find:

$$\nabla \times \left(\frac{\mathbf{J}(\bar{r}')}{|\bar{r} - \bar{r}'|} \right) = \frac{\mathbf{J}(\bar{r}') \times (\bar{r} - \bar{r}')}{|\bar{r} - \bar{r}'|^3}$$

and therefore the magnetic flux density is:

$$\mathbf{B}(\bar{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\bar{r}') \times (\bar{r} - \bar{r}')}{|\bar{r} - \bar{r}'|^3} \, dv'$$

This is know as the **Biot-Savart Law** !

For a **surface** current $\mathbf{J}_s(\bar{\mathbf{r}})$, the Biot-Savart Law becomes:

$$\mathbf{B}(\bar{\mathbf{r}}) = \frac{\mu_0}{4\pi} \iint_S \frac{\mathbf{J}_s(\bar{\mathbf{r}}') \times (\bar{\mathbf{r}} - \bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|^3} \, ds'$$

and for **line** current I, flowing on contour C, the Biot-Savart Law is:

$$\mathbf{B}(\bar{\mathbf{r}}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{\overline{d\ell'} \times (\bar{\mathbf{r}} - \bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|^3}$$

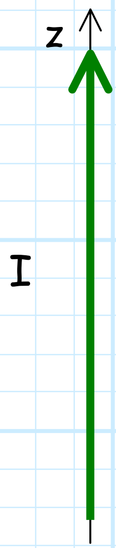
Note the contour C is **closed**. Do **you** know why?

*This is **dad-gum** outstanding! The Biot-Savart Law allows us to **directly** determine magnetic flux density* $\mathbf{B}(\bar{\mathbf{r}})$*, given some current density* $\mathbf{J}(\bar{\mathbf{r}})$*!*

Note that the Biot-Savart Law is therefore **analogous** to **Coloumb's Law** in Electrostatics (Do you see why?)!

Example: The Uniform, Infinite Line of Current

Consider electric current I flowing along the **z-axis** from $z=-\infty$ to $z=\infty$. What magnetic flux potential $\mathbf{B}(\bar{\mathrm{r}})$ is created by this current?



$$\overline{d\ell} = \hat{a}_z \, dz'$$

$$\begin{aligned}\bar{\mathrm{r}} &= x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z \\ &= \rho\cos\phi\,\hat{a}_x + \rho\sin\phi\,\hat{a}_y + z\,\hat{a}_z\end{aligned}$$

$$\bar{\mathrm{r}}' = z'\,\hat{a}_z \qquad (x'=0, y'=0)$$

$$\begin{aligned}|\bar{\mathrm{r}}-\bar{\mathrm{r}}'| &= \sqrt{\rho^2\cos^2\phi + \rho^2\sin^2\phi + (z-z')^2} \\ &= \sqrt{\rho^2 + (z-z')^2}\end{aligned}$$

We can determine the magnetic flux density by applying the **Biot-Savart Law**:

$$
\begin{aligned}
\mathbf{B}(\bar{\mathbf{r}}) &= \frac{\mu_0 I}{4\pi} \oint_C \frac{\overline{d\ell'} \times (\bar{\mathbf{r}} - \bar{\mathbf{r}}')}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|^3} \\
&= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{\hat{a}_z \times \left[\rho\cos\phi\, \hat{a}_x + \rho\sin\phi\, \hat{a}_y + (z - z')\, \hat{a}_z\right]}{\left[\rho^2 + (z - z')^2\right]^{3/2}}\, dz' \\
&= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho\cos\phi\, \hat{a}_y - \rho\sin\phi\, \hat{a}_x}{\left[\rho^2 + (z - z')^2\right]^{3/2}}\, dz' \\
&= \frac{\mu_0 I}{4\pi} \left(\rho\cos\phi\, \hat{a}_y - \rho\sin\phi\, \hat{a}_x\right) \int_{-\infty}^{\infty} \frac{du}{\left[\rho^2 + u^2\right]^{3/2}} \\
&= \frac{\mu_0 I}{4\pi} \left(\rho\, \hat{a}_\phi\right) \Bigg|_{-\infty}^{\infty} \frac{u}{\rho^2 \sqrt{\rho^2 + u^2}} \\
&= \frac{\mu_0 I}{4\pi} \left(\rho\, \hat{a}_\phi\right) \frac{2}{\rho^2} \\
&= \frac{\mu_0 I}{2\pi\rho}\, \hat{a}_\phi
\end{aligned}
$$

Therefore, the magnetic flux density **created** by a "wire" with current I flowing along the z-axis is:

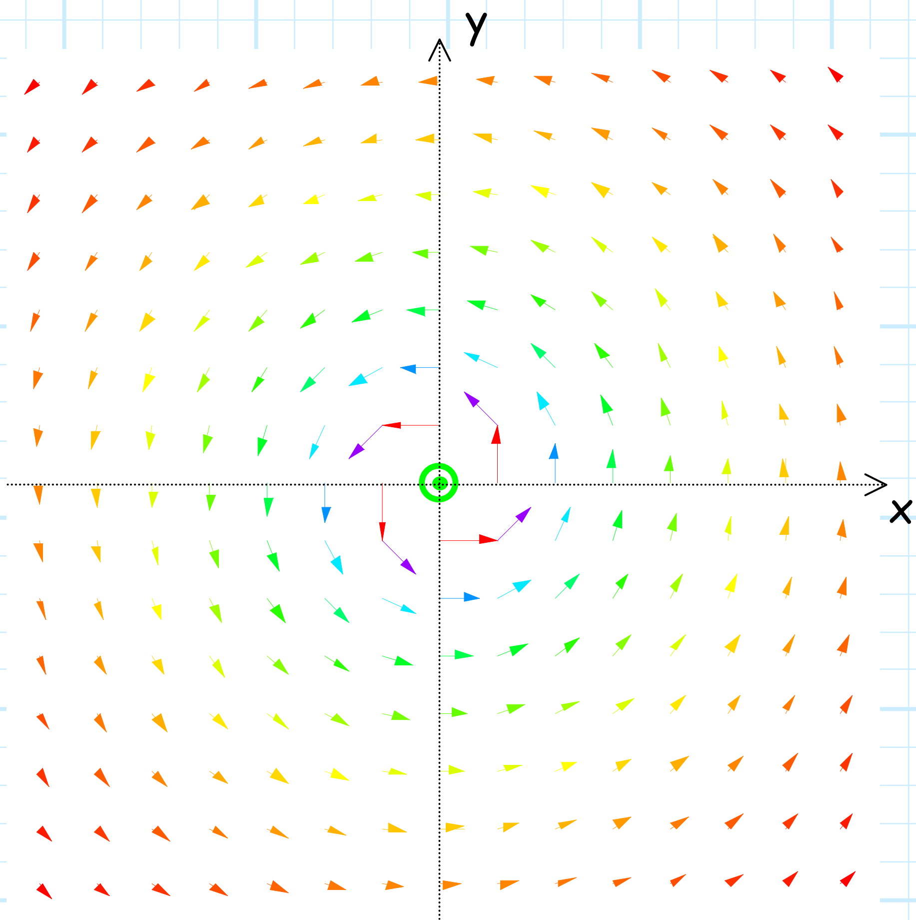
$$
\boxed{\mathbf{B}(\bar{\mathbf{r}}) = \frac{\mu_0 I}{2\pi\rho}\, \hat{a}_\phi}
$$

Think about what this expression tells us about magnetic flux density:

* The magnitude of $\mathbf{B}(\bar{r})$ is proportional to $1/\rho$, therefore magnetic flux density **diminishes** as we move farther from "wire".

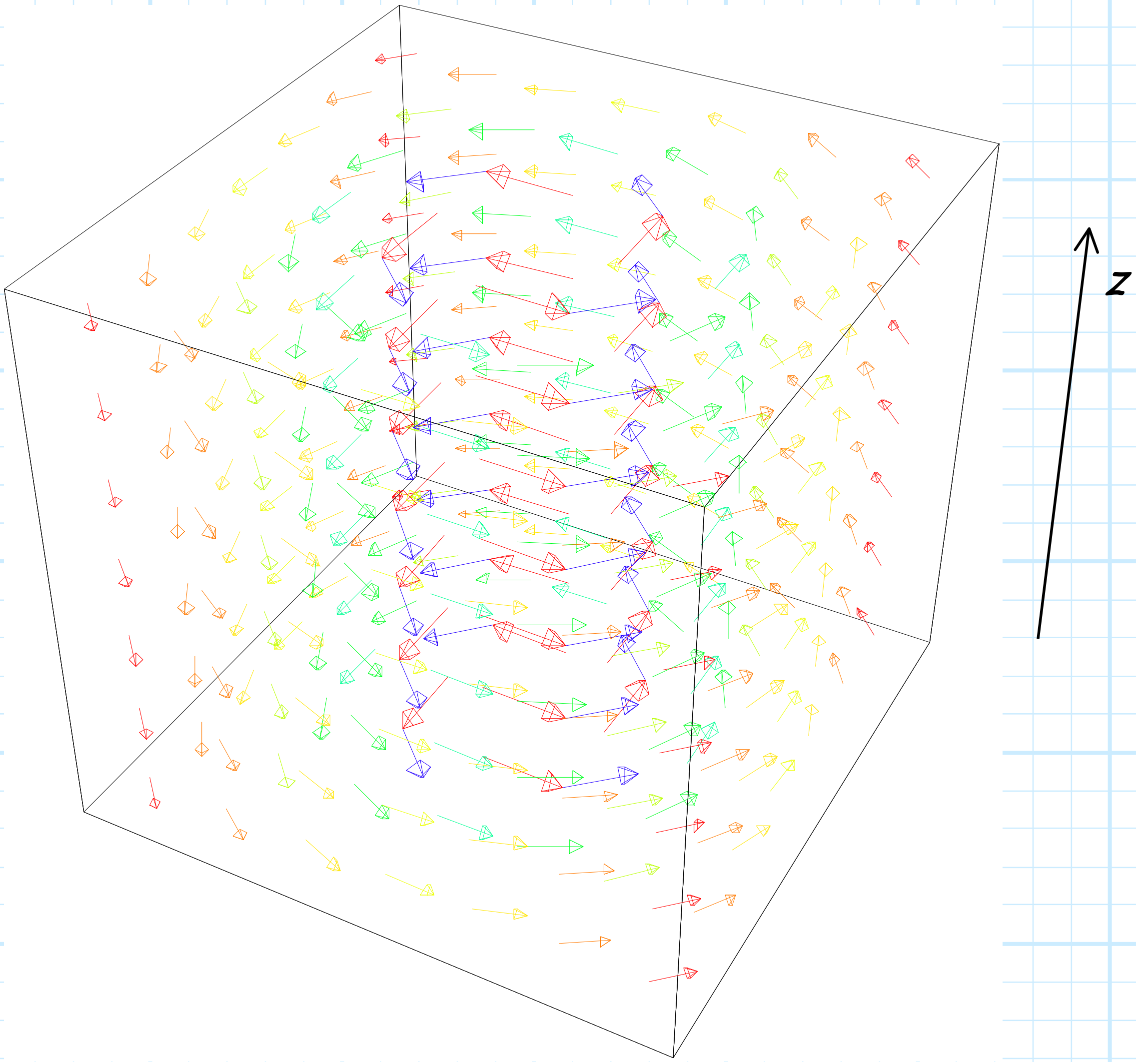
* The direction of $\mathbf{B}(\bar{r})$ is $\hat{a}_\phi$. In other words, the magnetic flux density points in the direction **around** the wire.

Plot of vector field $\mathbf{B}(\bar{r})$ on the x-y plane, resulting from current I flowing along the z-axis



= current I flowing out of this page.

Or, plotting in 3-D:

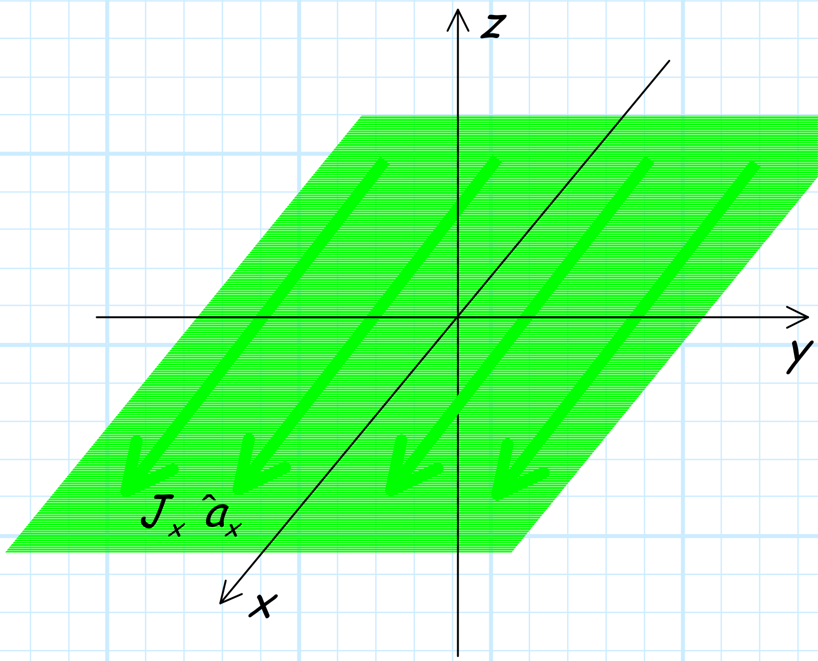


B-Field from an Infinite Sheet of Current

Consider now an **infinite sheet** of current, lying on the $z = 0$ plane. Say the surface current density on this sheet has a value:

$$\mathbf{J}_s(\bar{\mathbf{r}}) = J_x \, \hat{a}_x$$

meaning that the current density at every point on the surface has the same magnitude, and flows in the $\hat{a}_x$ direction.



Using the Biot-Savart Law, we find that the magnetic flux density produced by this **infinite** current sheet is:

$$\mathbf{B}(\bar{r}) = \begin{cases} -\dfrac{\mu_0 J_x}{2}\,\hat{a}_y & z > 0 \\ \\ \dfrac{\mu_0 J_x}{2}\,\hat{a}_y & z < 0 \end{cases}$$

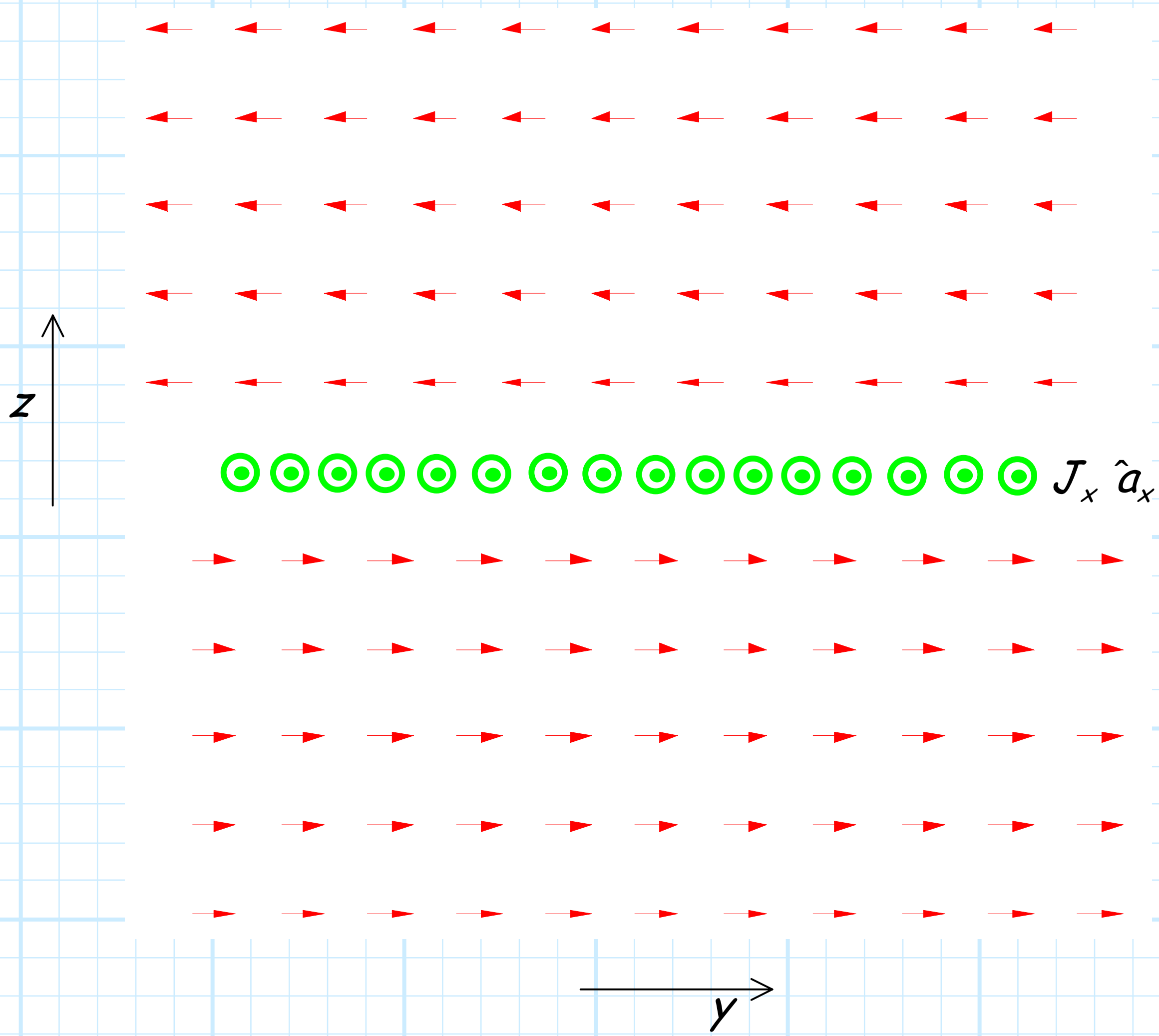
Think about what this expression is telling us.

* The magnitude of this magnetic flux density is a **constant**. In other words, $\mathbf{B}(\bar{r})$ is **just** as large a million miles from the infinite current sheet as it is 1 millimeter from the current sheet!

* The **direction** of the magnetic flux density in the $-\hat{a}_y$ direction **above** the current sheet, but points in the **opposite** direction (i.e., $\hat{a}_y$) **below** it.

* The direction of the magnetic flux density is **orthogonal** to the direction of current flow $\hat{a}_x$.

Plotting the vector field $\mathbf{B}(\bar{r})$ along the *y-z* plane, we find:


z
$J_x \hat{a}_x$
y