# E.M. Wave Propagation in Free-Space

Recall Maxwell's Equations for free-space:

$$\boxed{\begin{gathered}\nabla \times \bar{E}(\bar{r},t) = -\mu_0 \frac{\partial \bar{H}(\bar{r},t)}{\partial t} \\ \nabla \times \bar{H}(\bar{r},t) = \epsilon_0 \frac{\partial \bar{E}(\bar{r},t)}{\partial t} + \bar{J}(r,t) \\ \nabla \cdot \bar{E}(\bar{r},t) = \frac{\rho(\bar{r},t)}{\epsilon_0} \qquad \nabla \cdot \bar{H}(\bar{r},t) = 0\end{gathered}}$$

In a region with <u>no</u> sources, (i.e., $\bar{J}(\bar{r},t)=0$ & $\rho(\bar{r},t)=0$) Maxwell's Equations become:

$$\boxed{\begin{array}{ll} \nabla \times \overline{E}(\overline{r},t) = -\mu_0 \dfrac{\partial \overline{H}(\overline{r},t)}{\partial t} & \\ \nabla \times \overline{H}(\overline{r},t) = \epsilon_0 \dfrac{\partial \overline{E}(\overline{r},t)}{\partial t} & \\ \nabla \cdot \overline{E}(\overline{r},t) = 0 & \nabla \cdot \overline{H}(\overline{r},t) = 0 \end{array}}$$

**Q:** But, What does this have to do with e.m. wave propagation??

**A:** Be patient!

First, take the <u>curl</u> of <u>Faraday's Law</u>:

① $$\nabla \times \nabla \times \overline{E}(\overline{r},t) = \mu_0 \frac{\partial \nabla \times \overline{H}(\overline{r},t)}{t}$$

Hey! We <u>know</u> $\nabla \times \overline{H}(\overline{r},t) = \epsilon_0 \dfrac{\partial E(\overline{r},t)}{\partial t}$

Inserting this in equation ①, we get:

② $\nabla \times \nabla \times \bar{E}(\bar{r},t) = -\mu_0 \epsilon_0 \dfrac{\partial^2 \bar{E}(\bar{r},t)}{\partial t^2}$

From a mathematical identity (trust me), we know:

$$\nabla \times \nabla \times \bar{E}(\bar{r},t) = \nabla(\nabla \cdot \bar{E}(\bar{r},t)) - \nabla^2 \bar{E}(\bar{r},t)$$

But! Recall $\nabla \cdot \bar{E}(\bar{r},t) = 0$

Gauss's Law in free space!

$\therefore \nabla \times \nabla \times \bar{E}(\bar{r},t) = -\nabla^2 \bar{E}(\bar{r},t)$

So, we can rewrite ② as:

$$\boxed{\nabla^2 \bar{E}(\bar{r},t) - \mu_0 \epsilon_0 \frac{\partial^2 \bar{E}(\bar{r},t)}{\partial t^2} = 0}$$

This equation is called the vector wave equation.

In free-space, $\bar{E}(\bar{r},t)$ must satisfy this differential equation.

In other words, <u>only</u> electric fields over space and time that <u>satisfy</u> the wave equation can physical <u>exist</u> in free-space!

$\therefore$ Most functions $\bar{E}(\bar{r},t)$ are <u>not</u> physically possible.

We find that $\mu_0 \epsilon_0 = \dfrac{1}{c^2}$, where

$c = 3 \times 10^8$ m/s = velocity of "light" in free-space!

$\therefore$ We can write the vector wave equation as:

$$\boxed{\nabla^2 \bar{E}(\bar{r},t) - \frac{1}{c^2} \frac{\partial \bar{E}(\bar{r},t)}{\partial t} = 0}$$

<u>Two</u> solutions of this equation are:

1) Plane wave

$$\boxed{\bar{E}(\bar{r},t) = \bar{e}\,\exp\left[j\omega\left(x/c - t\right)\right]}$$

Where $\bar{e}$ is a vector constant describing the orientation of the electric field.

* For this case, $\bar{e}\cdot\hat{x}$ <u>must</u> equal 0 (i.e., $\bar{e}\cdot\hat{x} = 0$) for the wave equation to be satisfied.

* $\hat{x}$ is the direction of propagation of this plane wave, so $\bar{e}$ is perpendicular to the direction of propagation.

* The value $\omega$ indicates that this wave is sinusoidal, with frequency $\omega$ radians/sec.

2) Spherical Wave

$$\boxed{\bar{E}(\bar{r},t) = \bar{e}\,\frac{\exp\left[j\omega\left(r/c - t\right)\right]}{r}}$$

where $\bar{e}\cdot\hat{r} = 0$.

* This is a <u>spherical</u> wave, propagating away from the origin ($r=0$).

* The direction of the field vector $\bar{e}$ is perpendicular to direction of propagation $\hat{r}$ (i.e., $\bar{e}\cdot\hat{r}=0$).

{ There are many other solutions to the vector wave equation! }