The Poynting Vector

Recall that plane waves and spherical waves are electro-magnetic waves.

In other words, they consist of both electric and magnetic fields!

Q: Just what is the magnetic field $\bar{H}(\bar{r})$??

A: Use Faraday's Law!

i.e. $$\nabla \times \bar{E}(\bar{r},t) = -\mu_0 \frac{\partial \bar{H}(\bar{r},t)}{\partial t}$$

If $$\bar{E}(\bar{r},t) = \bar{e} \exp\left[j(kx-\omega t)\right]$$

then we find:

$$\bar{H}(\bar{r},t) = \bar{h} \frac{1}{\eta} \exp\left[j(kx-\omega t)\right]$$

where $|\bar{e}| = |\bar{h}|$, $\bar{h} \cdot \bar{e} = 0$, $\bar{h} \cdot \hat{x} = 0$

and

$$\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\,\Omega$$

This value, $\eta = 377\,\Omega$, is known as the characteristic impedance of free-space.

Q: Why $377\,\underline{\underline{\Omega}}$ (i.e., impedance)?

A: Note $\eta = \dfrac{|\bar{E}|}{|\bar{H}|}$!

Recall the unit of an electric field is V/m and for magnetic field A/m.

$\therefore \dfrac{|\bar{E}|}{|\bar{H}|}$ has units of $\dfrac{V/m}{A/m} = \dfrac{V}{A} = \underline{\underline{\text{Ohms}}}$

$\Rightarrow \dfrac{|\bar{E}|}{|\bar{H}|} =$ impedance !

Now, let's examine the Poynting Vector:

$$\overline{W}(\bar{r}) = \frac{1}{2} \operatorname{Re}\left\{ \overline{E}(\bar{r},t) \times \overline{H}^*(\bar{r},t) \right\}$$

where $*$ denotes the complex conjugate and $\times$ denotes the vector cross product.

For our plane-wave example, we find:

$$\overline{W}(\bar{r}) = \frac{1}{2} \left| \overline{E}(\bar{r},t) \right| \left| \overline{H}(r,t) \right| \hat{x}$$

$$= \frac{\left| \overline{E}(\bar{r},t) \right|^2}{2\eta} \hat{x}$$

$$= \frac{|\bar{e}|^2}{2\eta} \hat{x}$$

Q: What the heck does this mean?

A: Check out the units of $\overline{W}(\bar{r})$!

$$\left| \overline{E}(\bar{r},t) \right| \Rightarrow \frac{\text{Volts}}{m} \qquad \left| \overline{H}(\bar{r},t) \right| \Rightarrow \frac{\text{Amps}}{m}$$

$\therefore$ $|\overline{W}|$ has units of $\frac{V}{m} \cdot \frac{A}{m} = \frac{VA}{m^2}$

But, (volts)(amps) = <u>Watts</u> $\Rightarrow$ power!

$\therefore$ $|\overline{W}|$ has units of $\frac{\text{watts}}{m^2}$

$\Rightarrow$ Power / unit area

$\therefore$

1) The magnitude of $\overline{W}(\bar{r})$ is the <u>power</u> <u>density</u> of the e.m. wave!

2) The direction of $\overline{W}(\bar{r})$ (i.e., $\hat{x}$ for our example) describes the direction of power flow.

* Note for a <u>plane</u>-<u>wave</u>, the power density is <u>constant</u>, i.e., $|\overline{W}(\bar{r})|$ is independent of x.

$$|\overline{W}(\bar{r})| = \frac{|\bar{e}|^2}{2\eta}\,\hat{x}$$

* But, this is not necessarily true for all propagating e.m. waves!

* For example, the Poynting vector for a spherical wave is:

$$\bar{W}(\bar{r}) = \frac{|\bar{e}|^2}{2\eta} \frac{\hat{r}}{r^2}$$

So the power density decreases as r^{-2}

i.e., $|\bar{W}(\bar{r})| \propto \frac{1}{r^2}$ for spherical wave