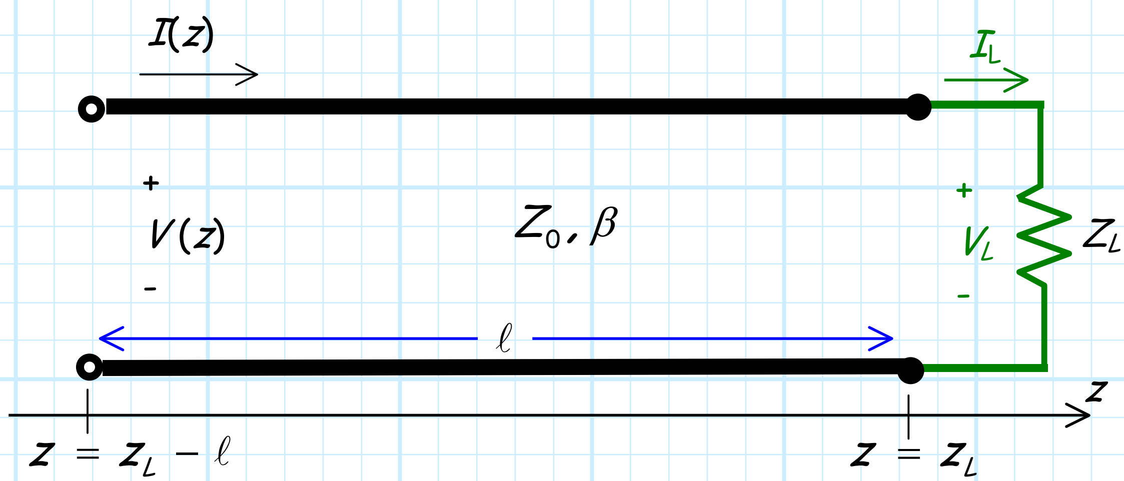
The Terminated, Lossless Transmission Line

Now let's **attach** something to our transmission line. Consider a **lossless** line, length ℓ, terminated with a **load** Z_L.



Q: *What is the* ***current*** *and* ***voltage*** *at each and* ***every*** *point on the transmission line (i.e., what is $I(z)$ and $V(z)$ for* ***all*** *points z where $z_L - \ell \leq z \leq z_L$?)?*

A: To find out, we must apply **boundary conditions**!

In other words, at the **end** of the transmission line ($z = z_L$)—where the load is **attached**—we have **many** requirements that **all** must be satisfied!

1. To begin with, the voltage and current ($I(z=z_L)$ and $V(z=z_L)$) must be consistent with a valid **transmission line solution**:

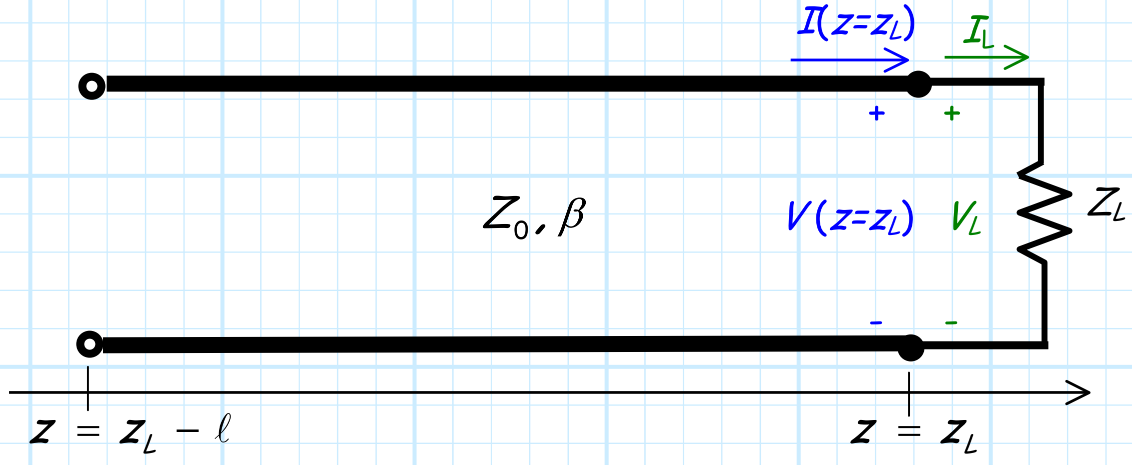
$$\begin{aligned} V(z=z_L) &= V^+(z=z_L) + V^-(z=z_L) \\ &= V_0^+ e^{-j\beta z_L} + V_0^- e^{+j\beta z_L} \end{aligned}$$

$$\begin{aligned} I(z=z_L) &= \frac{V_0^+(z=z_L)}{Z_0} - \frac{V_0^-(z=z_L)}{Z_0} \\ &= \frac{V_0^+}{Z_0} e^{-j\beta z_L} - \frac{V_0^-}{Z_0} e^{+j\beta z_L} \end{aligned}$$

2. Likewise, the load voltage and current must be related by **Ohm's law**:

$$V_L = Z_L \, I_L$$

3. Most importantly, we recognize that the values $I(z=z_L)$, $V(z=z_L)$ and I_L, V_L are **not** independent, but in fact are strictly related by **Kirchoff's Laws**!



From KVL and KCL we find these requirements:

$$V(z = z_L) = V_L$$

$$I(z = z_L) = I_L$$

These are the **boundary conditions** for **this** particular problem.

→ **Careful**! Different transmission line problems lead to **different** boundary conditions—**you** must access each problem **individually** and **independently**!

Combining these equations and boundary conditions, we find that:

$$V_L = Z_L \, I_L$$

$$V(z = z_L) = Z_L \, I(z = z_L)$$

$$V^+(z = z_L) + V^-(z = z_L) = \frac{Z_L}{Z_0}\left(V^+(z = z_L) - V^-(z = z_L)\right)$$

Rearranging, we can conclude:

$$\frac{V^-(z = z_L)}{V^+(z = z_L)} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Q: *Hey wait as second! We earlier defined $V^-(z)/V^+(z)$ as* ***reflection coefficient*** *$\Gamma(z)$. How does this relate to the expression* ***above****?*

A: Recall that $\Gamma(z)$ is a **function** of transmission line position *z*. The value $V^-(z=z_L)/V^+(z=z_L)$ is simply the value of function $\Gamma(z)$ **evaluated** at $z=z_L$ (i.e., evaluated at the **end** of the line):

$$\frac{V^-(z=z_L)}{V^+(z=z_L)} = \Gamma(z=z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

This value is of **fundamental** importance for the terminated transmission line problem, so we provide it with its **own** special symbol (Γ_L) !

$$\boxed{\Gamma_L \doteq \Gamma(z=z_L) = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

Q: *Wait! We* ***earlier*** *determined that:*

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0}$$

so it would seem that:

$$\Gamma_L = \Gamma(z=z_L) = \frac{Z(z=z_L) - Z_0}{Z(z=z_L) + Z_0}$$

Which *expression is correct??*

A: They **both** are! It is evident that the two expressions:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad \text{and} \qquad \Gamma_L = \frac{Z(z = z_L) - Z_0}{Z(z = z_L) + Z_0}$$

are **equal** if:

$$Z(z = z_L) = Z_L$$

And since we know that from **Ohm's Law**:

$$Z_L = \frac{V_L}{I_L}$$

and from **Kirchoff's Laws**:

$$\frac{V_L}{I_L} = \frac{V(z = z_L)}{I(z = z_L)}$$

and that **line impedance** is:

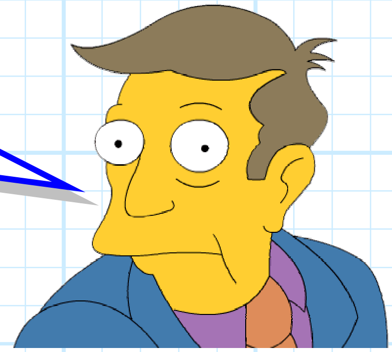
$$\frac{V(z = z_L)}{I(z = z_L)} = Z(z = z_L)$$

we find it apparent that the **line impedance** at the **end** of the transmission line is **equal** to the **load impedance**:

$$Z(z = z_L) = Z_L$$

The above expression is essentially **another** expression of the **boundary condition** applied at the **end** of the transmission line.

Q: *I'm confused! Just what are were we trying to accomplish in this handout?*



A: We are trying to find $V(z)$ and $I(z)$ when a lossless transmission line is terminated by a load Z_L!

We can now determine the value of V_0^- in terms of V_0^+. Since:

$$\Gamma_L = \frac{V^-(z=z_L)}{V^+(z=z_L)} = \frac{V_0^- e^{+j\beta z_L}}{V_0^+ e^{-j\beta z_L}}$$

We find:

$$V_0^- = e^{-2j\beta z_L}\,\Gamma_L\, V_0^+$$

And therefore we find:

$$V^-(z) = \left(e^{-2j\beta z_L}\,\Gamma_L\, V_0^+\right) e^{+j\beta z}$$

$$V(z) = V_0^+\left[e^{-j\beta z} + \left(e^{-2j\beta z_L}\,\Gamma_L\right) e^{+j\beta z}\right]$$

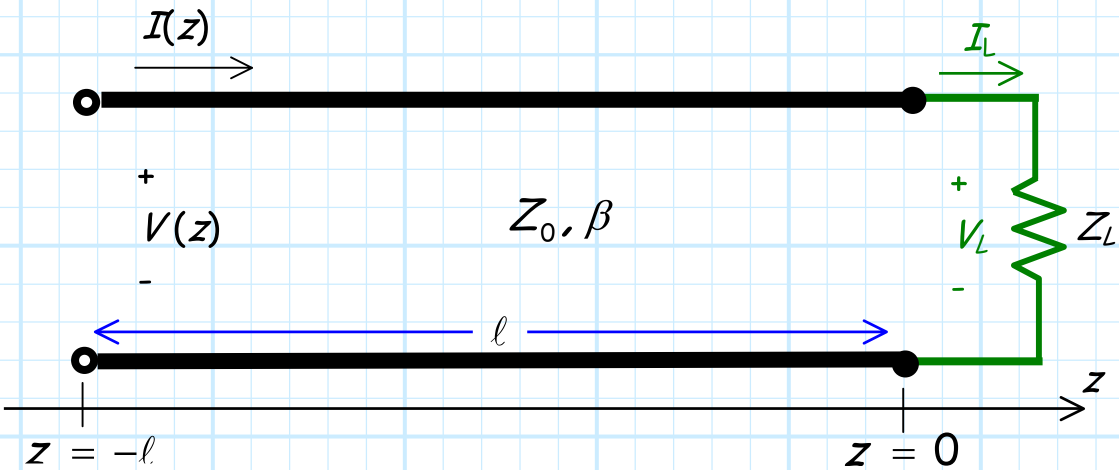
$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} - \left(e^{-2j\beta z_L}\,\Gamma_L\right) e^{+j\beta z}\right]$$

where:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$z_L = 0$

Now, we can further **simplify** our analysis by **arbitrarily** assigning the end point z_L a **zero** value (i.e., $z_L = 0$):



If the load is located at z=0 (i.e., **if** $z_L = 0$), we find that:

$$\begin{aligned} V(z=0) &= V^+(z=0) + V^-(z=0) \\ &= V_0^+ e^{-j\beta(0)} + V_0^- e^{+j\beta(0)} \\ &= V_0^+ + V_0^- \end{aligned}$$

$$\begin{aligned} I(z=0) &= \frac{V_0^+(z=0)}{Z_0} - \frac{V_0^-(z=0)}{Z_0} \\ &= \frac{V_0^+}{Z_0} e^{-j\beta(0)} - \frac{V_0^-}{Z_0} e^{+j\beta(0)} \\ &= \frac{V_0^+ - V_0^-}{Z_0} \end{aligned}$$

$$Z(z=0) = Z_0 \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \right)$$

Likewise, it is apparent that **if** $z_L = 0$, Γ_L and Γ_0 are the same:

$$\Gamma_L = \Gamma(z = z_L) = \frac{V^-(z=0)}{V^+(z=0)} = \frac{V_0^-}{V_0^+} = \Gamma_0$$

Therefore:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_0$$

Thus, we can write the line current and voltage simply as:

$$V(z) = V_0^+\left[e^{-j\beta z} + \Gamma_L\, e^{+j\beta z}\right]$$

$$\left[\text{for } z_L = 0\right]$$

$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} - \Gamma_L\, e^{+j\beta z}\right]$$

Q: *But, how do we determine V_0^+ ??*

A: We require a **second** boundary condition to determine V_0^+. The only boundary left is at the **other end** of the transmission line. Typically, a **source** of some sort is located there. This makes physical sense, as something must generate the **incident** wave !