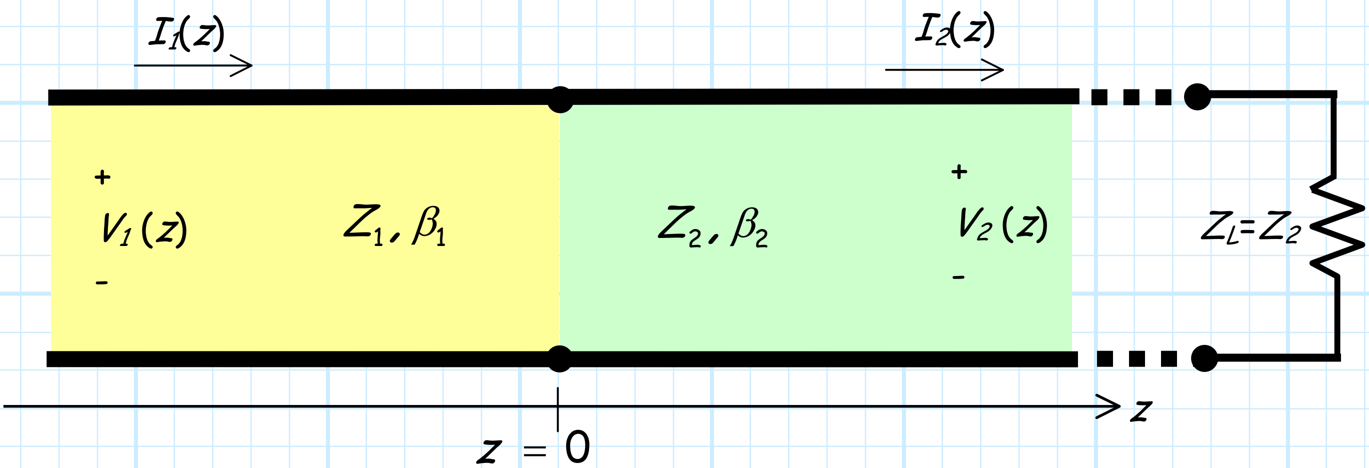
The Transmission Coefficient T

Consider this circuit:



I.E., a transmission line with characteristic impedance Z_1 **transitions** to a **different** transmission line at location z =0. This second transmission line has different **characteristic impedance** Z_2 $(Z_1 \neq Z_2)$. This second line is **terminated** with a load $Z_L = Z_2$ (i.e., the second line is **matched**).

Q: *What is the voltage and current along each of these two transmission lines? More specifically, what are* V_{01}^+, V_{01}^-, V_{02}^+ and V_{02}^- *??*

A: Since a source has not been specified, we can only determine V_{01}^-, V_{02}^+ and V_{02}^- in terms of complex constant V_{01}^+. To accomplish this, we must apply a **boundary condition** at z =0!

$z < 0$

We know that the voltage along the **first** transmission line is:

$$V_1(z) = V_{01}^{+} e^{-j\beta_1 z} + V_{01}^{-} e^{+j\beta_1 z} \qquad [\text{for } z < 0]$$

while the **current** along that same line is described as:

$$I_1(z) = \frac{V_{01}^{+}}{Z_1} e^{-j\beta_1 z} - \frac{V_{01}^{-}}{Z_1} e^{+j\beta_1 z} \qquad [\text{for } z < 0]$$

$z > 0$

We likewise know that the voltage along the **second** transmission line is:

$$V_2(z) = V_{02}^{+} e^{-j\beta_2 z} + V_{02}^{-} e^{+j\beta_2 z} \qquad [\text{for } z > 0]$$

while the **current** along that same line is described as:

$$I_2(z) = \frac{V_{02}^{+}}{Z_2} e^{-j\beta_2 z} - \frac{V_{02}^{-}}{Z_2} e^{+j\beta_2 z} \qquad [\text{for } z > 0]$$

Moreover, since the second line is terminated in a **matched load**, we know that the **reflected** wave from this load must be zero:

$$V_2^{-}(z) = V_{02}^{-} e^{-j\beta_2 z} = 0$$

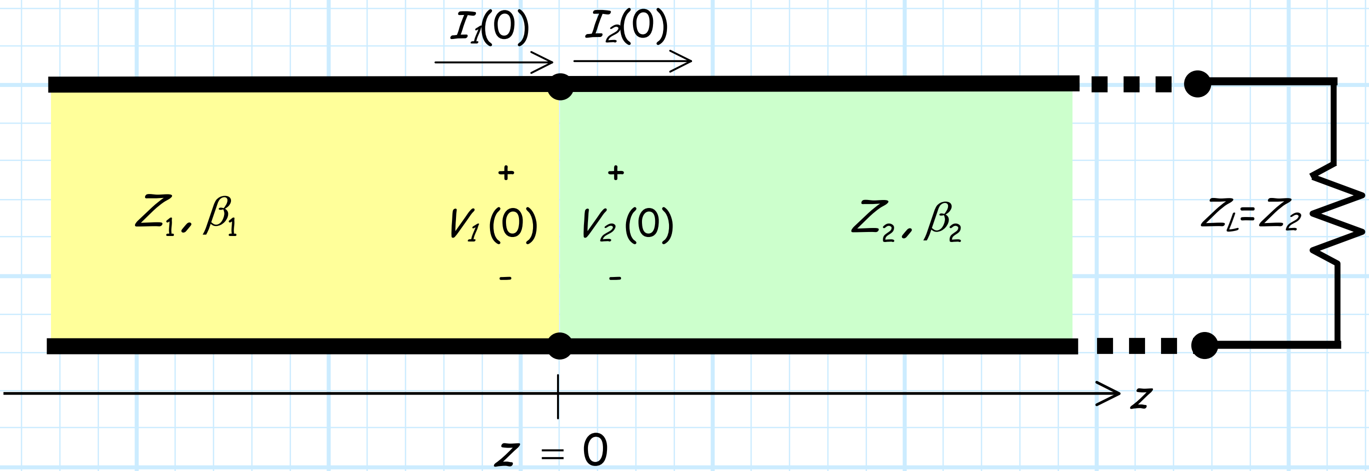
The voltage and current along the **second** transmission line is thus simply:

$$V_2(z) = V_2^+(z) = V_{02}^+ \, e^{-j\beta_2 z} \qquad [\text{for } z > 0]$$

$$I_2(z) = I_2^+(z) = \frac{V_{02}^+}{Z_2} e^{-j\beta_2 z} \qquad [\text{for } z > 0]$$

z=0

At the location where these two transmission lines meet, the current and voltage expressions each **must** satisfy some specific **boundary conditions**:



The **first** boundary condition comes from **KVL**, and states that:

$$V_1(z=0) = V_2(z=0)$$

$$V_{01}^+ \; e^{-j\beta_1(0)} + V_{01}^- \; e^{+j\beta_1(0)} = V_{02}^+ \, e^{-j\beta_2(0)}$$

$$V_{01}^+ + V_{01}^- = V_{02}^+$$

while the **second** boundary condition comes from **KCL**, and states that:

$$I_1(z=0) = I_2(z=0)$$

$$\frac{V_{01}^+}{Z_1} e^{-j\beta_1(0)} - \frac{V_{01}^-}{Z_1} e^{+j\beta_1(0)} = \frac{V_{02}^+}{Z_2} e^{-j\beta_2(0)}$$

$$\frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} = \frac{V_{02}^+}{Z_2}$$

We now have **two** equations and **two** unknowns (V_{01}^- and V_{02}^+)! We can **solve** for each in terms of V_{01}^+ (i.e., the **incident** wave).

From the **first** boundary condition we can state:

$$V_{01}^- = V_{02}^+ - V_{01}^+$$

Inserting this into the **second** boundary condition, we find an expression involving **only** V_{02}^+ and V_{01}^+:

$$\frac{V_{01}^+}{Z_1} - \frac{V_{01}^-}{Z_1} = \frac{V_{02}^+}{Z_2}$$

$$\frac{V_{01}^+}{Z_1} - \frac{V_{02}^+ - V_{01}^+}{Z_1} = \frac{V_{02}^+}{Z_2}$$

$$\frac{2V_{01}^+}{Z_1} = \frac{V_{02}^+}{Z_2} + \frac{V_{02}^+}{Z_1}$$

Solving this expression, we find:

$$V_{02}^+ = \left(\frac{2Z_2}{Z_1 + Z_2}\right) V_{01}^+$$

We can therefore define a **transmission coefficient**, which relates V_{02}^{+} to V_{01}^{+}:

$$T_0 \doteq \frac{V_{02}^{+}}{V_{01}^{+}} = \frac{2Z_2}{Z_1 + Z_2}$$

Meaning that $V_{02}^{+} = T\, V_{01}^{+}$, and thus:

$$V_2(z) = V_2^{+}(z) = T\, V_{01}^{+}\, e^{-j\beta_2 z} \qquad \left[\text{for } z > 0\right]$$

We can **likewise** determine the constant V_{01}^{-} in terms of V_{01}^{+}. We again start with the **first** boundary condition, from which we concluded:

$$V_{02}^{+} = V_{01}^{+} + V_{01}^{-}$$

We can insert this into the **second** boundary condition, and determine an expression involving V_{01}^{-} and V_{01}^{+} **only**:

$$\frac{V_{01}^{+}}{Z_1} - \frac{V_{01}^{-}}{Z_1} = \frac{V_{02}^{+}}{Z_2}$$

$$\frac{V_{01}^{+}}{Z_1} - \frac{V_{01}^{-}}{Z_1} = \frac{V_{01}^{+} + V_{01}^{-}}{Z_2}$$

$$\left(\frac{1}{Z_1} - \frac{1}{Z_2}\right) V_{01}^{+} = \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) V_{01}^{-}$$

Solving this expression, we find:

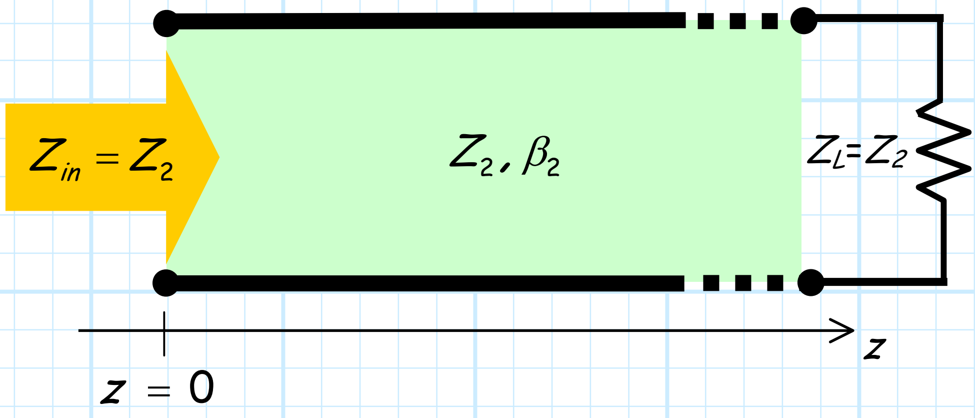
$$V_{01}^{-} = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1} \right) V_{01}^{+}$$

We can therefore define a **reflection coefficient**, which relates V_{01}^{-} to V_{01}^{+}:

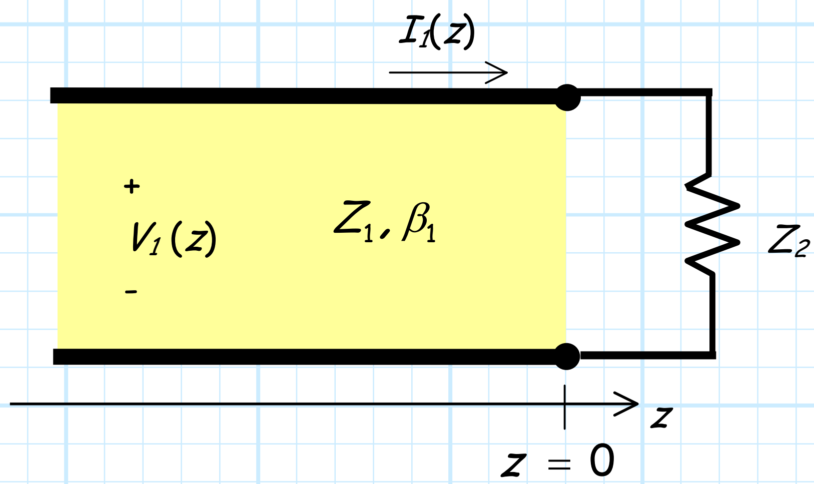
$$\Gamma_0 \doteq \frac{V_{01}^{-}}{V_{01}^{+}} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

This result should **not** surprise us!

Note that because the **second** transmission line is **matched**, its **input** impedance is equal to Z_1:



and thus we can **equivalently** write the entire circuit as:



We have already analyzed **this** circuit! We know that:

$$
\begin{aligned}
V_{01}^- &= \Gamma_L V_{01}^+ \\
&= \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right) V_{01}^+
\end{aligned}
$$

Which is **exactly** the same result as we determined earlier!

The values of the reflection coefficient Γ_0 and the transmission coefficient T_0 are **not** independent, but in fact are directly **related**. Recall the **first** boundary expressed was:

$$V_{01}^+ + V_{01}^- = V_{02}^+$$

Dividing this by V_{01}^+:

$$1 + \frac{V_{01}^-}{V_{01}^+} = \frac{V_{02}^+}{V_{01}^+}$$

Since $\Gamma_0 = V_{01}^{-}/V_{01}^{+}$ and $T_0 = V_{02}^{+}/V_{01}^{+}$:

$$1 + \Gamma_0 = T_0$$

Note the result $T_0 = 1 + \Gamma_0$ is true for **this** particular circuit, and therefore is **not** a universally valid expression for two-port networks!