Algorithm Efficiency & Sorting

- Algorithm efficiency
- Big-O notation
- Searching algorithms
- Sorting algorithms

Overview

- Writing programs to solve problem consists of a large number of decisions
  - how to represent aspects of the problem for solution
  - which of several approaches to a given solution component to use
- If several algorithms are available for solving a given problem, the developer must choose among them
- If several ADTs can be used to represent a given set of problem data
  - which ADT should be used?
  - how will ADT choice affect algorithm choice?

Overview – 2

- If a given ADT (i.e. stack or queue) is attractive as part of a solution
- How will the ADT implement affect the program's:
  - correctness and performance?
- Several goals must be balanced by a developer in producing a solution to a problem
  - correctness, clarity, and efficient use of computer resources to produce the best performance
- How is solution performance best measured?
  - *time* and *space*

Overview – 3

- The order of importance is, generally,
  - correctness
  - efficiency
  - clarity
- Clarity of expression is qualitative and somewhat dependent on perception by the reader
  - developer salary costs dominate many software projects
  - time efficiency of understanding code written by others can thus have a significant monetary implication
- Focus of this chapter is *execution efficiency*
  - mostly, run-time (some times, memory space)

Measuring Algorithmic Efficiency

- Analysis of algorithms
  - provides tools for contrasting the efficiency of different methods of solution
- Comparison of algorithms
  - should focus on *significant* differences in efficiency
  - should not consider reductions in computing costs due to clever coding tricks
- Difficult to compare programs instead of algorithms
  - how are the algorithms coded?
  - what computer should you use?
  - what data should the programs use?

Analyzing Algorithmic Cost

- Viewed abstractly, an algorithm is a sequence of steps
  - Algorithm A { S1; S2; .... Sm1; Sm }
- The total cost of the algorithm will thus, obviously, be the total cost of the algorithm's *m* steps
  - assume we have a function giving cost of each statement

$$Cost\ (S_i) = execution\ cost\ of\ S_i,\ for\text{-}all\ i,\ 1 \leq i \leq m$$

- Total cost of the algorithm's m steps would thus be:

$$Cost\ (A) = \sum_{i=1}^{m} Cost\ (Si)$$

Analyzing Algorithmic Cost – 2

- However, an algorithm can be applied to a wide variety of problems and data sizes
  - so we want a cost function for the algorithm A that takes the data set size *n* into account

    $$Cost\ (A, n) = \sum_1^n \left(\sum_1^m (Cost\ (S_i))\right)$$
- Several factors complicate things
  - conditional statements: cost of evaluating condition and branch taken
  - loops: cost is sum of each of its iterations
  - recursion: may require solving a recurrence equation

Analyzing Algorithmic Cost – 3

- Do not attempt to accumulate a precise prediction for program execution time, because
  - far too many complicating factors: compiler instructions output, variation with specific data sets, target hardware speed
- Provides an approximation, an *order of magnitude* estimate, that permits fair comparison of one algorithm's behavior against that of another

Analyzing Algorithmic Cost – 4

- Various behavior bounds are of interest
  - best case, average case, worst case
- Worst-case analysis
  - A determination of the maximum amount of time that an algorithm requires to solve problems of size n
- Average-case analysis
  - A determination of the average amount of time that an algorithm requires to solve problems of size n
- Best-case analysis
  - A determination of the minimum amount of time that an algorithm requires to solve problems of size n

Analyzing Algorithmic Cost – 5

- Complexity measures can be calculated in terms of
  - T(n): time complexity and S(n): space complexity
- Basic model of computation used
  - sequential computer (one statement at a time)
  - all data require same amount of storage in memory
  - each datum in memory can be accessed in constant time
  - each basic operation can be executed in constant time
- Note that all of these assumptions are incorrect!
  - good for this purpose
- Calculations we want are order of magnitude

Example – Linked List Traversal

- Assumptions

  C_1 = cost of assign.

  C_2 = cost of compare

  C_3 = cost of write

```
Node *cur = head;        // assignment op
   while (cur != NULL) // comparisons op
   cout << cur→item
         << endl;        // write op
   cur→next;             // assignment op
}
```

- Consider the number of operations for n items

$$T(n) = (n+1)C_1 + (n+1)C_2 + nC_3$$
$$= (C_1+C_2+C_3)n + (C_1+C_2) = K_1 n + K_2$$

- Says, algorithm is of linear complexity
  – work done grows linearly with n but also involves constants

Example – Sequential Search

- Number of comparisons

  $T_B(n) = 1$ (or 3?)

  $T_w(n) = n$

  $T_A(n) = (n+1)/2$

- In general, what developers worry about the most is that this is O(n) algorithm
  - more precise analysis is nice but rarely influences algorithmic decision

```
Seq_Search(A: array, key: integer);
    i = 1;
    while i ≤ n and A[i] ≠ key do
        i = i + 1
    endwhile;
    if i ≤ n
        then return(i)
        else return(0)
    endif;
end Sequential_Search;
```

Bounding Functions

- To provide a guaranteed bound on how much work is involved in applying an algorithm **A** to **n** items
  - we find a bounding function f(n) such that

$$T(n) \leq f(n), \forall\, n$$

- It is often easier to satisfy a less stringent constraint by finding an elementary function f(n) such that

$$T(n) \leq k * f(n), for\ sufficiently\ large\ n$$

- This is denoted by the asymptotic **big-O** notation
- Algorithm A is O(n) says
  - that complexity of A is no worse than k*n as n grows sufficiently large

Asymptotic Upper Bound

- Defn: A function f is positive if $f(n) > 0, \forall\, n > 0$
- Defn: Given a positive function f(n), then

$$f(n) = O\big(g(n)\big)$$

iff there exist constants k > 0 and n_0 > 0 such that

$$f(n) \leq k * g(n), \forall\, n > n_0$$

- Thus, g(n) is an asymptotic bounding function for the work done by the algorithm
- k and n_0 can be *any* constants
  - can lead to unsatisfactory conclusions if they are very large and a developer's data set is relatively small

Asymptotic Upper Bound – 2

- Example: show that: $2n^2 - 3n + 10 = O(n^2)$
- Observe that

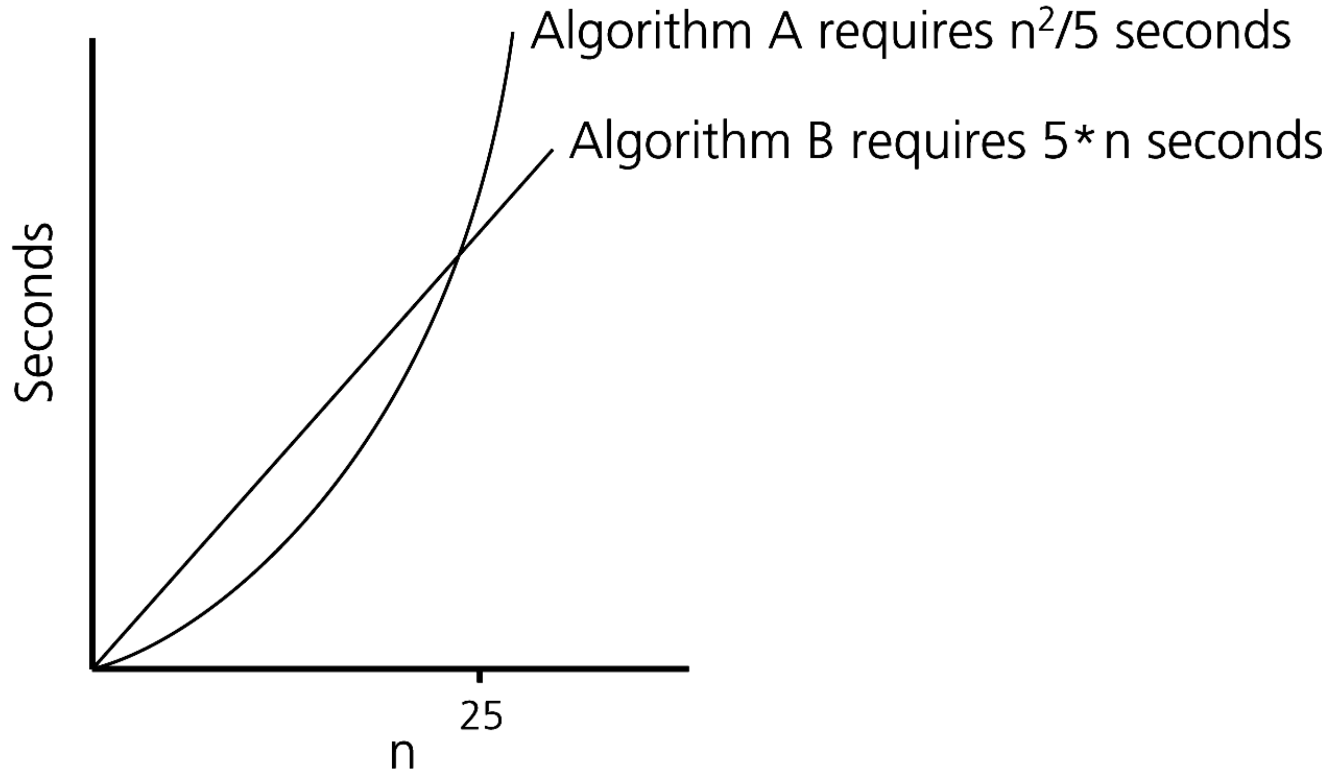
$$2n^2 - 3n + 10 \leq 2n^2 + 10, n > 1$$
$$2n^2 - 3n + 10 \leq 2n^2 + 10, n^2 \; n > 1$$
$$2n^2 - 3n + 10 \leq 12n^2, n > 1$$

- Thus, expression is $O(n^2)$ for k = 12 and $n_0 > 1$ (also k = 3 and $n_0 > 1$, BTW)
  - algorithm efficiency is typically a concern for large problems only
- Then, O(f(n)) information helps choose a set of final candidates and direct measurement helps final choice

Algorithm Growth Rates

- An algorithm's time requirements can be measured as a function of the problem size
  - Number of nodes in a linked list
  - Size of an array
  - Number of items in a stack
  - Number of disks in the Towers of Hanoi problem

Algorithm Growth Rates – 2



- Algorithm A requires time proportional to n^2
- Algorithm B requires time proportional to n

Algorithm Growth Rates – 3

- An algorithm's growth rate enables comparison of one algorithm with another
- Example
  - if, algorithm A requires time proportional to $n^{2,}$ and algorithm B requires time proportional to n
  - algorithm B is faster than algorithm A
  - n^2 and n are growth-rate functions
  - Algorithm A is $O(n^2)$ - order n^2
  - Algorithm B is $O(n)$ - order n
- Growth-rate function f(n)
  - mathematical function used to specify an algorithm's order in terms of the size of the problem

Order-of-Magnitude Analysis and Big O Notation

(a)

Function	n = 10	100	1,000	10,000	100,000	1,000,000
1	1	1	1	1	1	1
$\log_2 n$	3	6	9	13	16	19
n	10	10^2	10^3	10^4	10^5	10^6
$n * \log_2 n$	30	664	9,965	10^5	10^6	10^7
n^2	10^2	10^4	10^6	10^8	10^{10}	10^{12}
n^3	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}
2^n	10^3	10^{30}	10^{301}	$10^{3,010}$	$10^{30,103}$	$10^{301,030}$

Figure 9-3a A comparison of growth-rate functions: (a) in tabular form

Order-of-Magnitude Analysis and Big O Notation

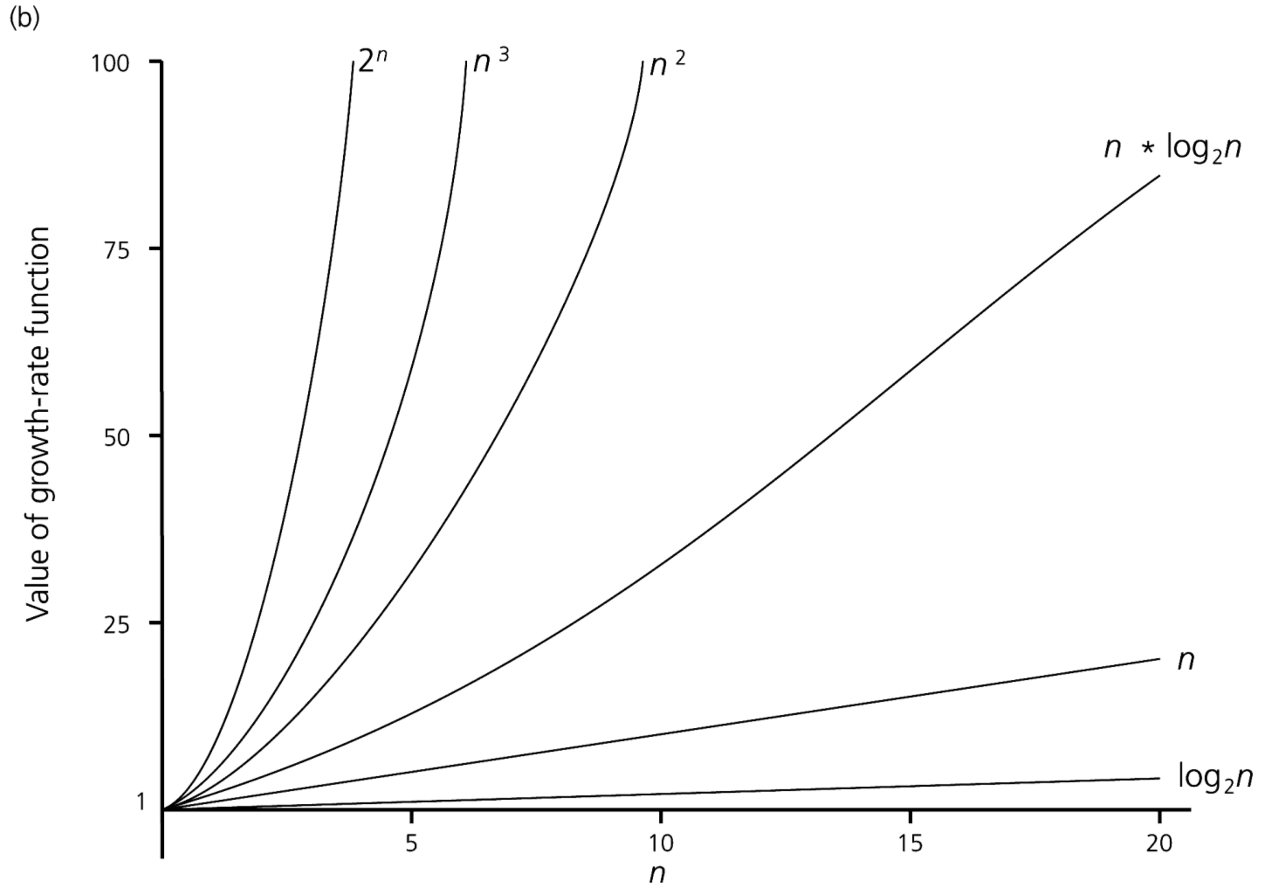


Figure 9-3b A comparison of growth-rate functions: (b) in graphical form

Order-of-Magnitude Analysis and Big O Notation

- Order of growth of some common functions
  - $O(C) < O(\log(n)) < O(n) < O(n * \log(n)) < O(n^2) < O(n^3) < O(2^n) < O(3^n) < O(n!) < O(n^n)$
- Properties of growth-rate functions
  - O(n3 + 3n) is O(n3): ignore low-order terms
  - O(5 f(n)) = O(f(n)): ignore multiplicative constant in the high-order term
  - O(f(n)) + O(g(n)) = O(f(n) + g(n))

Keeping Your Perspective

- Only significant differences in efficiency are interesting
- Frequency of operations
  - when choosing an ADT's implementation, consider how frequently particular ADT operations occur in a given application
  - however, some seldom-used but critical operations must be efficient

Keeping Your Perspective

- If the problem size is always small, you can probably ignore an algorithm's efficiency
  - order-of-magnitude analysis focuses on large problems
- Weigh the trade-offs between an algorithm's time requirements and its memory requirements
- Compare algorithms for both style and efficiency

Sequential Search

- Sequential search
  - look at each item in the data collection in turn
  - stop when the desired item is found, or the end of the data is reached

```
int search(const int a[ ], int number_used, int target) {
	int index = 0; bool found = false;
	while ((!found) && (index < number_used)) {
		if (target == a[index])
			found = true;
		else
			 Index++;
	 }
	if (found)   return index;
	else    return 1;
}
```

Efficiency of Sequential Search

- Worst case: O(n)
  - key value not present, we search the entire list to prove failure
- Average case: O(n)
  - all positions for the key being equally likely
- Best case: O(1)
  - key value happens to be first

The Efficiency of Searching Algorithms

- Binary search of a sorted array
  - Strategy
    - Repeatedly divide the array in half
    - Determine which half could contain the item, and discard the other half
  - Efficiency
    - Worst case: $O(\log_2 n)$
    - For large arrays, the binary search has an enormous advantage over a sequential search
      - At most 20 comparisons to search an array of one million items

Sorting Algorithms and Their Efficiency

- Sorting
  - A process that organizes a collection of data into either ascending or descending order
  - The sort key is the data item that we consider when sorting a data collection
- Sorting algorithm types
  - comparison based
    - bubble sort, insertion sort, quick sort, etc.
  - address calculation
    - radix sort

Sorting Algorithms and Their Efficiency

- Categories of sorting algorithms
  - An internal sort
    - Requires that the collection of data fit entirely in the computer's main memory
  - An external sort
    - The collection of data will not fit in the computer's main memory all at once, but must reside in secondary storage

Selection Sort

- Strategy
  - Place the largest (or smallest) item in its correct place
  - Place the next largest (or next smallest) item in its correct place, and so on
- Algorithm

```
for index=0 to size-2 {
    select min/max element from among A[index], …, A[size-1];
    swap(A[index], min);
}
```

- Analysis
  - worst case: O(n2), average case: O(n2)
  - does not depend on the initial arrangement of the data

Selection Sort

Shaded elements are selected;
boldface elements are in order.

Initial array:	29	10	14	37	13
After 1st swap:	29	10	14	13	**37**
After 2nd swap:	13	10	14	**29**	**37**
After 3rd swap:	13	10	**14**	**29**	**37**
After 4th swap:	**10**	**13**	**14**	**29**	**37**

Bubble Sort

- Strategy
  - compare adjacent elements and exchange them if they are out of order
    - moves the largest (or smallest) elements to the end of the array
  - repeat this process
    - eventually sorts the array into ascending (or descending) order
- Analysis: worst case: O(n2), best case: O(n)

Bubble Sort – algorithm

```
for i = 1 to size -- 1   do
    for index = 1 to size -- i do
        if A[index] < A[index1]
            swap(A[index], A[index1]);
    endfor;
endfor;
```

Bubble Sort

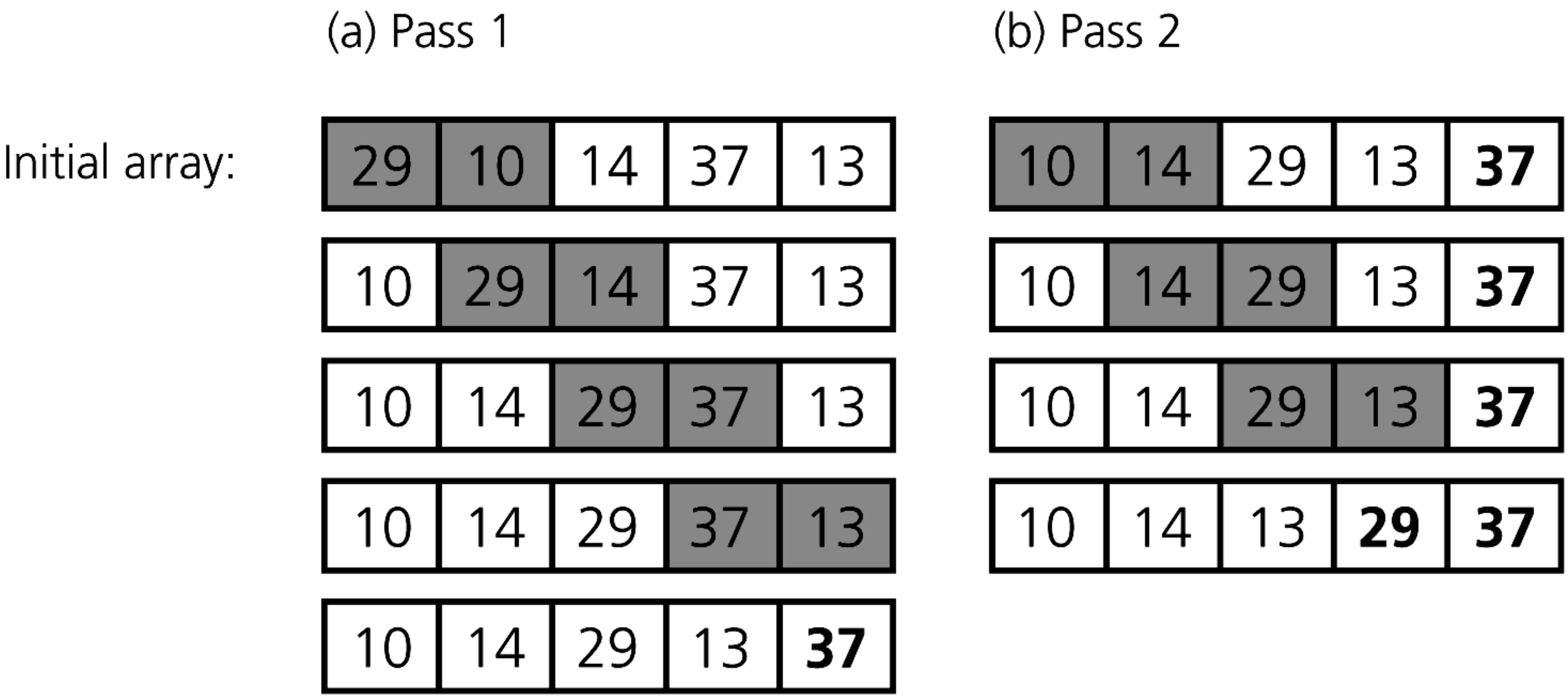


Figure 9-5

The first two passes of a bubble sort of an array of five integers: (a) pass 1; (b) pass 2

Insertion Sort

- Strategy
  - Partition array in two regions: sorted and unsorted
    - initially, entire array is in unsorted region
    - take each item from the unsorted region and insert it into its correct position in the sorted region
    - each *pass* shrinks unsorted region by 1 and grows sorted region by 1
- Analysis
  - Worst case: O(n2)
    - Appropriate for small arrays due to its simplicity
    - Prohibitively inefficient for large arrays

Insertion Sort

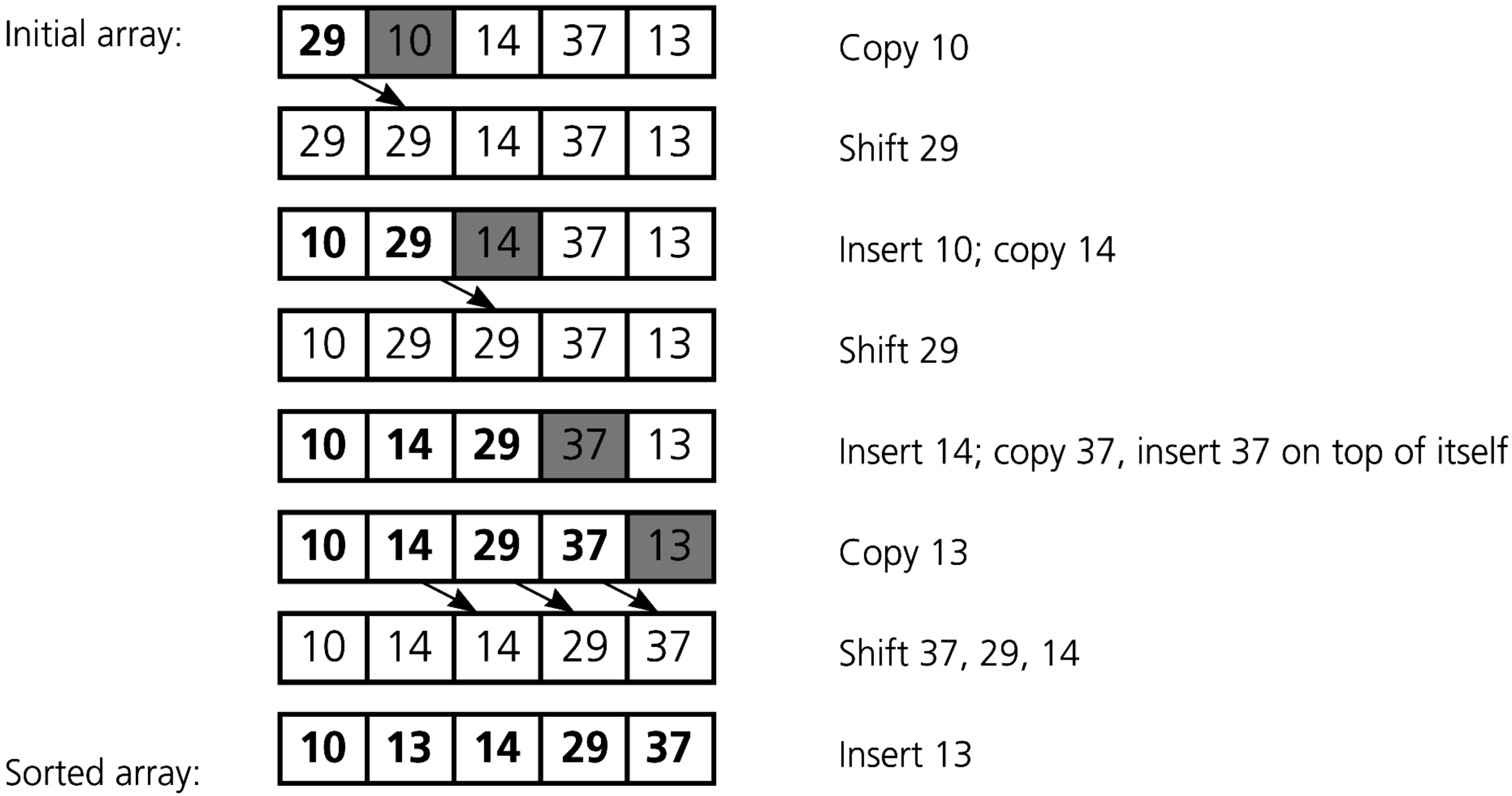


Figure 9-7 An insertion sort of an array of five integers.

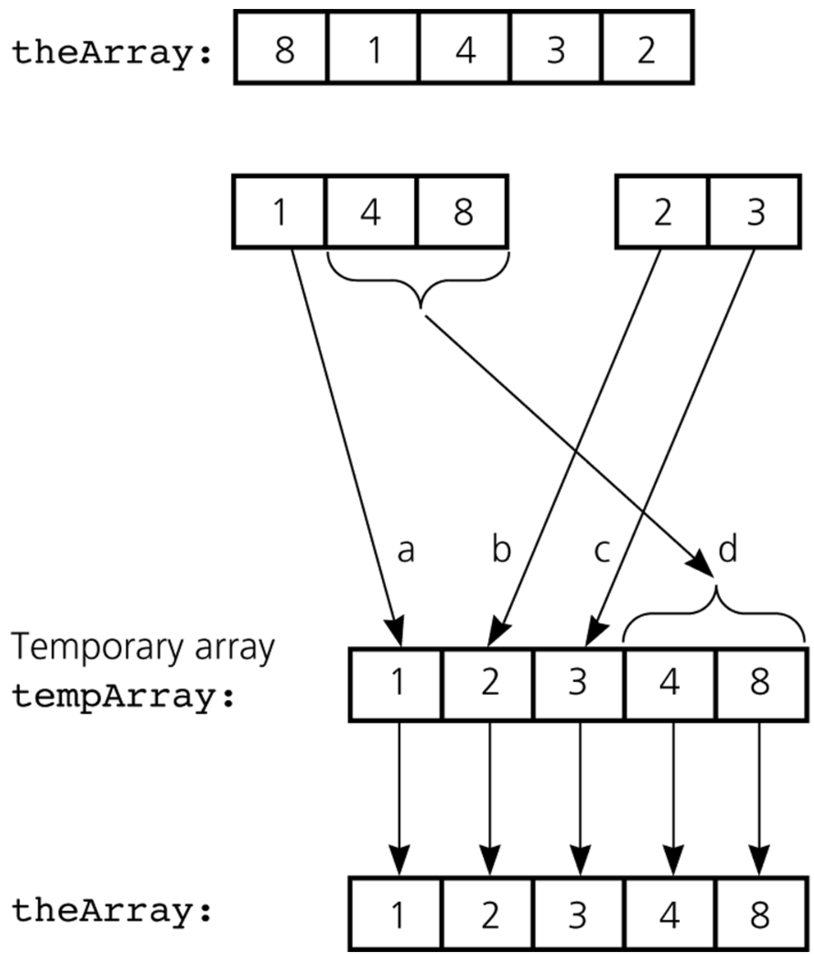
Mergesort

- A recursive sorting algorithm
- Performance is independent of the initial order of the array items
- Strategy
  - divide an array into halves
  - sort each half
  - merge the sorted halves into one sorted array
  - divide-and-conquer approach

Mergesort – Algorithm

```
mergeSort(A,first,last) {
        if (first < last) {
                mid = (first + last)/2;
                mergeSort(A, first, mid);
                mergeSort(A, mid+1, last);
                merge(A, first, mid, last)
        }
}
```

Mergesort



Divide the array in half

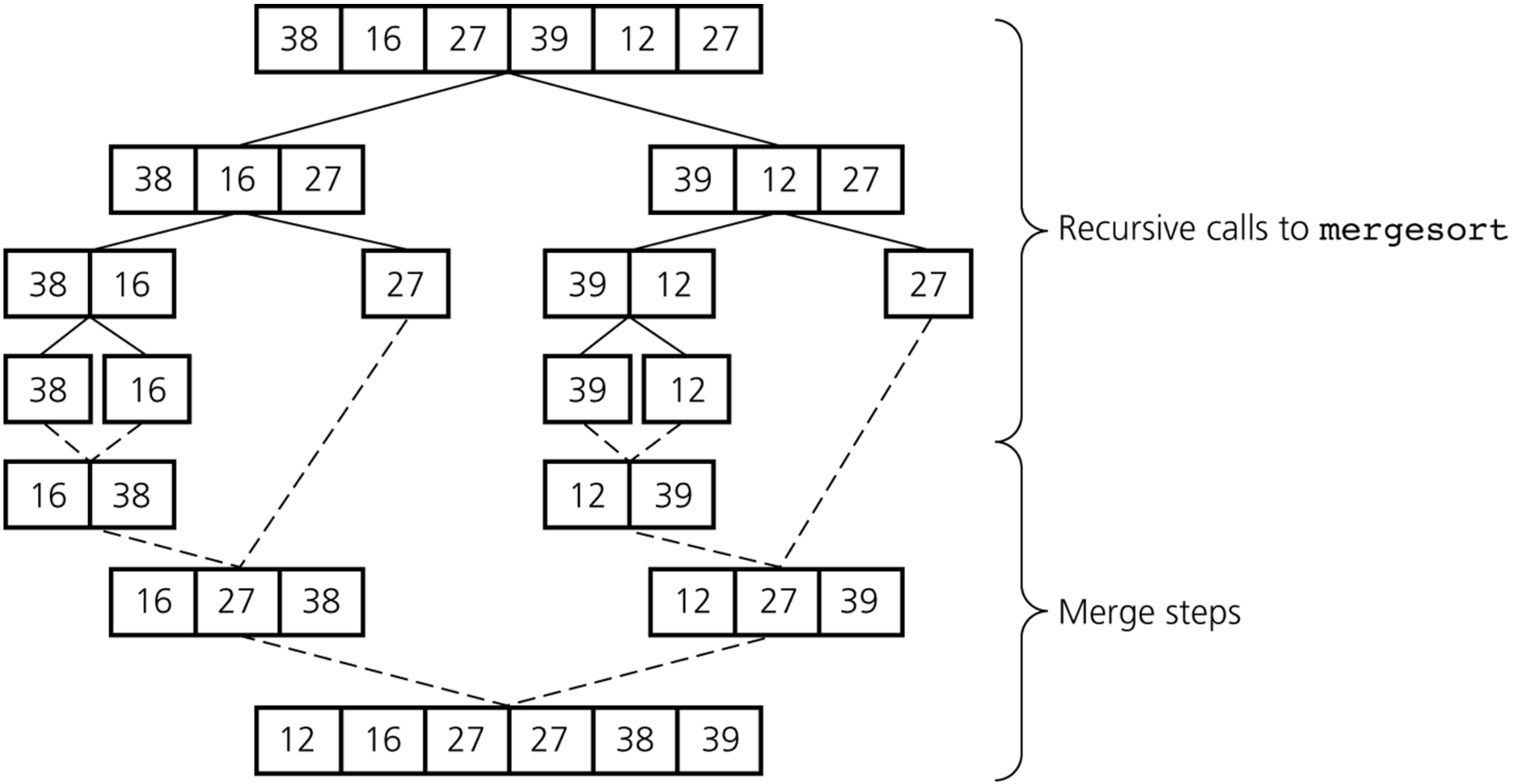
Sort the halves

Merge the halves:

a. 1 < 2, so move 1 from left half to `tempArray`
b. 4 > 2, so move 2 from right half to `tempArray`
c. 4 > 3, so move 3 from right half to `tempArray`
d. Right half is finished, so move rest of left half to `tempArray`

Copy temporary array back into original array

Mergesort



38 16 27 39 12 27
38 16 27
39 12 27
38 16
27
39 12
27
38
16
39
12
16 38
12 39
16 27 38
12 27 39
12 16 27 27 38 39
Recursive calls to mergesort
Merge steps

Mergesort – Properties

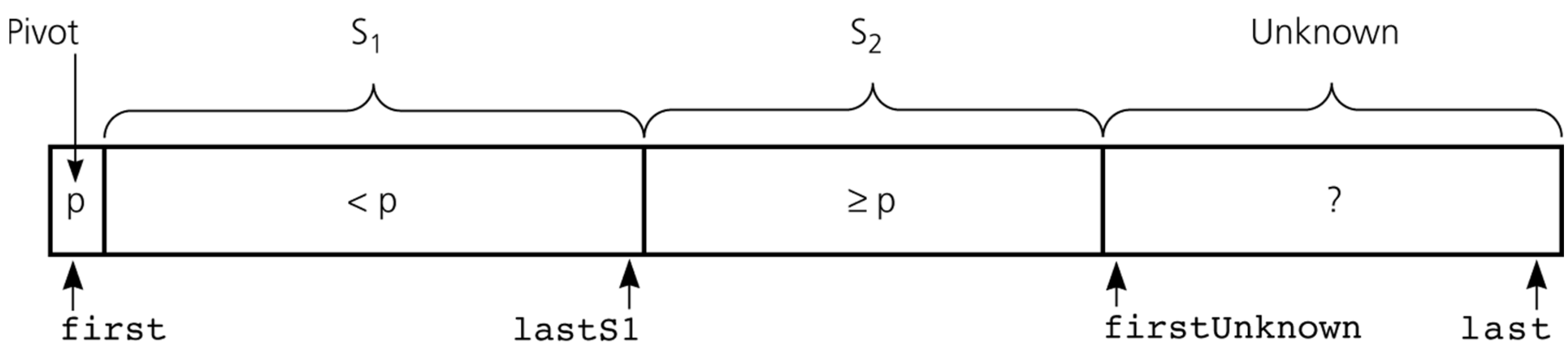
- Needs a temporary array into which to copy elements during merging
  - doubles space requirement
- Mergesort is *stable*
  - items with equal key values appear in the same order in the output array as in the input
- Advantage
  - mergesort is an extremely fast algorithm
- Analysis: worst / average case: O(n * log2n)

Quicksort

- A recursive divide-and-conquer algorithm
  - given a linear data structure A with n records
  - divide A into sub-structures S_1 and S_2
  - sort S_1 and S_2 recursively
- Algorithm
  - Base case: if |S|==1, S is already sorted
  - Recursive case:
    - divide A around a pivot value P into S_1 and S_2 , such that all elements of S_1<=P and all elements of S_2>=P
    - recursively sort S1 and S2 in place

Quicksort

- Partition()
  - (a) scans array, (b) chooses a pivot, (c) divides A around pivot, (d) returns pivot index
  - Invariant: items in S_1 are all less than pivot, and items in S_2 are all greater than or equal to pivot
- Quicksort()
  - partitions A, sorts S_1 and S_2 recursively



Quicksort – Pivot Partitioning

- Pivot selection and array partition are fundamental work of algorithm
- Pivot selection
  - perfect value: median of A[ ]
    - sort required to determine median (oops!)
    - approximation: If |A| > N, N==3 or N==5, use median of N
  - Heuristic approaches used instead
    - Choose A[first] OR A[last] OR A[mid] (mid = (first+last)/2) OR Random element
    - heuristics equivalent if contents of A[ ] randomly arranged

Quicksort – Pivot Partitioning Example

A= [5,8,3,7,4,2,1,6], first =0, last =7

- 1. A[first]: pivot = 5
- 2. A[last]: pivot = 6
- 3. A[mid]: mid =(0+7)/2=3, pivot = 7
- 4. A[random()]: any key might be chosen
- 5. A[medianof3]: median(A[first], A[mid], A[last]) is
- median(5,7,6) = 6
- ● Note that the median determination is itself a sort,
- but only of a fixed number of items, which is thus
- still O(1)
- ● Good pivot selection
- ● Computed in O(1) time and partitions A into
- roughly equal parts S1 and S2

Quicksort

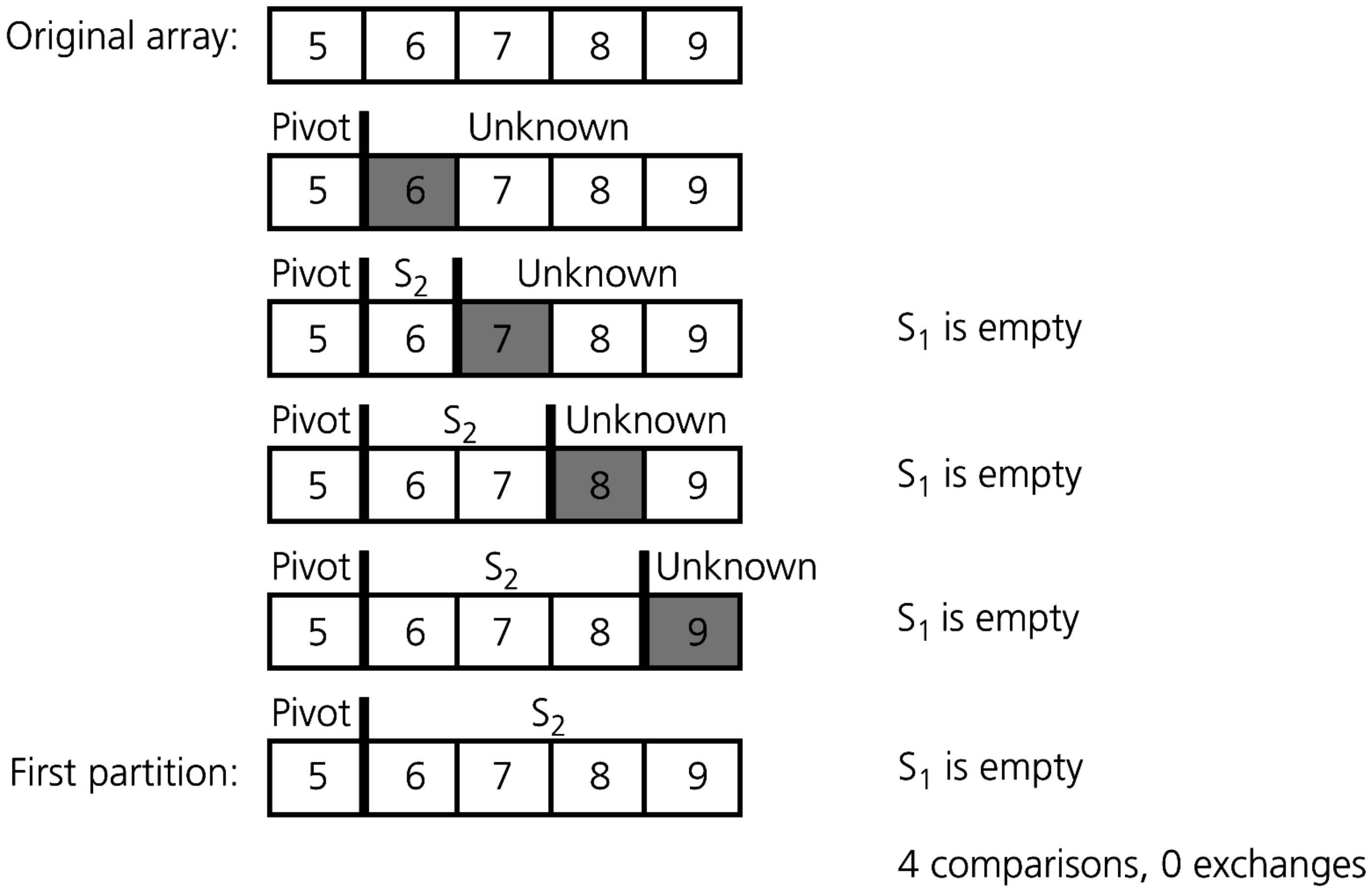


Figure 9-19 A worst-case partitioning with **`quicksort`**

Quicksort

- Analysis
  - Average case: O(n * log2n)
  - Worst case: O(n2)
    - When the array is already sorted and the smallest item is chosen as the pivot
  - Quicksort is usually extremely fast in practice
  - Even if the worst case occurs, quicksort's performance is acceptable for moderately large arrays

Radix Sort

- Strategy
  - Treats each data element as a character string
  - Repeatedly organizes the data into groups according to the ith character in each element
- Analysis
  - Radix sort is O(n)

Radix Sort

0123, 2154, 0222, 0004, 0283, 1560, 1061, 2150	Original integers
(156**0**, 215**0**) (106**1**) (022**2**) (012**3**, 028**3**) (215**4**, 000**4**)	Grouped by fourth digit
1560, 2150, 1061, 0222, 0123, 0283, 2154, 0004	Combined
(00**0**4) (02**2**2, 01**2**3) (21**5**0, 21**5**4) (15**6**0, 10**6**1) (02**8**3)	Grouped by third digit
0004, 0222, 0123, 2150, 2154, 1560, 1061, 0283	Combined
(0**0**04, 1**0**61) (0**1**23, 2**1**50, 2**1**54) (0**2**22, 0**2**83) (1**5**60)	Grouped by second digit
0004, 1061, 0123, 2150, 2154, 0222, 0283, 1560	Combined
(**0**004, **0**123, **0**222, **0**283) (**1**061, **1**560) (**2**150, **2**154)	Grouped by first digit
0004, 0123, 0222, 0283, 1061, 1560, 2150, 2154	Combined (sorted)

Figure 9-21 A radix sort of eight integers

A Comparison of Sorting Algorithms

	Worst case	Average case
Selection sort	n^2	n^2
Bubble sort	n^2	n^2
Insertion sort	n^2	n^2
Mergesort	$n * \log n$	$n * \log n$
Quicksort	n^2	$n * \log n$
Radix sort	n	n
Treesort rt	n^2	$n * \log n$
	$n * \log n$	$n * \log n$

Figure 9-22 Approximate growth rates of time required for eight sorting algorithms

The STL Sorting Algorithms

- Some sort functions in the STL library header <algorithm>
  - sort
    - Sorts a range of elements in ascending order by default
  - stable_sort
    - Sorts as above, but preserves original ordering of equivalent elements

The STL Sorting Algorithms

- partial_sort
  - Sorts a range of elements and places them at the beginning of the range
- nth_element
  - Partitions the elements of a range about the nth element
  - The two subranges are not sorted
- partition
  - Partitions the elements of a range according to a given predicate

Summary

- Order-of-magnitude analysis and Big O notation measure an algorithm's time requirement as a function of the problem size by using a growth-rate function
- To compare the efficiency of algorithms
  - Examine growth-rate functions when problems are large
  - Consider only significant differences in growth-rate functions

Summary

- Worst-case and average-case analyses
  - Worst-case analysis considers the maximum amount of work an algorithm will require on a problem of a given size
  - Average-case analysis considers the expected amount of work that an algorithm will require on a problem of a given size

Summary

- Order-of-magnitude analysis can be the basis of your choice of an ADT implementation
- Selection sort, bubble sort, and insertion sort are all $O(n^2)$ algorithms
- Quicksort and mergesort are two very fast recursive sorting algorithms