QUANTUM ALGORITHMS
EXAM 1

PROF. MATTHEW MOORE

DUE: 2020-04-09

Instructions

• Solutions must be typed. Submit solutions by email.

• Solutions will be graded based on correctness, quality, and presentation. Turn in something that you are proud of.

• You may make use of any non-human assistance — any book, the web (but do not ask for help online), etc. Solutions must be self-contained.

• You may ask me questions about the problems.

• You must submit a “draft” of your solution no later than 04-02.

We will consider a generalization of Grover’s algorithm. Suppose that we are given the following.

• $A$, a quantum circuit.

• Basis vector $|\text{start}\rangle$ and quantum state $|\text{end}\rangle$ (i.e. norm 1) with $A|\text{start}\rangle = |\text{end}\rangle$.

• $|\text{end}\rangle = |A\rangle + |B\rangle$ with $\langle A | B \rangle = 0$, $\langle A | A \rangle = a$, and $\langle B | B \rangle = b = 1 - a$.

Let us consider $|A\rangle$ as consisting of a superposition of “correct” outcomes of algorithm $A$.

Upon measuring $|\text{end}\rangle$, the probability of observing $|A\rangle$ is $a$.

We assume that we have a basis $(\psi_i)_{i\in I}$ such that $I = A \cup B$ and a function $\chi : I \to \{0, 1\}$ such that $\chi(A) = 1$ and $\chi(B) = 0$. Define

$$|\Psi(\alpha, \beta)\rangle = \alpha |A\rangle + \beta |B\rangle$$

and note that $|\Psi(1, 1)\rangle = |\text{end}\rangle$. Define $G = A \circ D_\beta \circ A^\dagger \circ D_A$ where $D_A$ and $D_\beta$ are reflection operators (use the phase shift $e^{i\theta}$ for both).

Give a careful analysis of the circuit $G \circ A$, addressing all of the points below.

• Draw the circuits for both reflection operators and also write out the operators for them (e.g. $D_0 = (e^{i\theta} - 1) |0\rangle \langle 0| + I$ as in the usual Grover’s algorithm).

• Calculate $G |\Psi(1, 1)\rangle$ and write your answer in the form $|\Psi(x, y)\rangle$ for some $x, y$ (you should specify their values).

• Suppose that $A$ works with probability $1/4$ (i.e. $a = 1/4$). Show that taking $\theta = \pi$ (as in the usual Grover’s circuit) makes $G \circ A$ acting on $|\text{start}\rangle$ exact.

• Show that when $A$ works with probability $1/2$ there is a choice of $\theta$ so that $G \circ A$ acting on $|\text{start}\rangle$ is exact. What is the value of $e^{i\theta}$ in this case?

• Show that when $A$ works with probability in the interval $[1/4, 1]$ there is a choice of $\theta$ so that $G \circ A$ acting on $|\text{start}\rangle$ is exact. Give a formula to determine it given $a$.

• When $A$ works with probability in the interval $(0, 1/4)$, is there a choice of $\theta$ that makes $G \circ A$ acting on $|\text{start}\rangle$ exact? Fully justify your answer.

(When a circuit $B$ gives the correct answer with probability 1, then $B$ is said to be exact.)