Abstracts

Eran Crockett

One of McNulty's easy problems

McNulty has (and others likely have) asked whether every finite dualizable algebra of finite signature is finitely based. I have found a dualizable algebra for which it is unknown whether or not it is finitely based.

Matt Valeriote and Ross Willard

Questions about Maltsev conditions

Recent results on the relative strength of specific Maltsev conditions, or the complexity of testing specific Maltsev conditions, lead to more questions.

Regarding the relative strength of Maltsev conditions:

(1) SD($\wedge$) is characterized at the level of locally finite varieties by a strong Maltsev condition [3, Theorem 2.8]. Is SD($\wedge$) equivalent, in general, to a strong Maltsev condition?

(2) Every locally finite SD($\wedge$) variety has an SD($\wedge$) term [1, Theorem 1.3], by which we mean an n-ary idempotent term t satisfying an $n \times n$ matrix equation $t[A] = t[B]$ where A has x's on its diagonal, B has y's on its diagonal, and the entries of A strictly below its diagonal are equal to the respective entries of B. Is this also true in the nonlocally finite case?

(3) Miroslav Olšák has shown us that Taylor varieties are characterized by a strong Maltsev condition. Hobby and McKenzie showed us a long time ago that locally finite Taylor varieties have a weak difference term. Is there a strong Maltsev condition characterizing "p is a weak difference term" at the level of locally finite varieties? What about the condition

$$\begin{aligned} s(x,x,x) &\approx t(x,x,x,x) \approx p(x,x,x) \approx x \\ s(x,x,y) &\approx s(x,y,x) \approx t(x,x,x,y) \approx t(x,x,y,x) \approx t(x,y,x,x) \\ s(y,x,x) &\approx t(y,x,x,x) \approx p(y,x,x) \approx p(x,x,y). \end{aligned}$$

Regarding the complexity of testing Maltsev conditions:

(1) Can every strong, idempotent linear Maltsev condition be tested in finite idempotent algebras in polynomial time? In particular, does this hold for testing $m(y,x,x) \approx m(x,y,x) \approx m(x,x,y) \approx y$?

(2) Are there *any* strong idempotent Maltsev conditions that are not equivalent to a linear condition but can be tested in finite idempotent algebras in polynomial time? What about the 2-semilattice condition $xx \approx x$, $xy \approx yx$, $x(xy) \approx xy$?

(3) Is there a polynomial-time algorithm to test if a finite (not necessarily idempotent) algebra has a Maltsev (or majority, or Pixley) term?

References

[1] Libor Barto, Marcin Kozik and David Stanovský, Mal'tsev conditions, lack of absorption, and solvability, *Algebra Universalis* **74** (2015), 185–206.

[2] Ralph Freese and Matthew A. Valeriote, On the complexity of some Maltsev conditions, *Internat. J. Algebra Comput.* **19** (2009), 41–77.

[3] Marcin Kozik, Andrei Krokhin, Matt Valeriote and Ross Willard, Characterizations of several Maltsev conditions, *Algebra Universalis* **73** (2015), 205–224.

Mirek Olšák

Local equational conditions in idempotent algebras

We say that a equational (strong Maltsev) condition is locally satisfied on a fixed set X if there are terms satisfying the condition, when only values on the set X are checked. We proved that local NU (on X) implies local Siggers term (on X). What other connections are there? Is there the weakest local equational condition?

Mirek Olšák

Weak single linear equations

It is known that the following is equivalent in finite algebra **A**:

(1) **A** contains a term satisfying a single nontrivial linear condition.
(2) **A** contains a term t satisfying $t(xyxzyz) = t(yxzxzy)$.
(3) **A** contains a term t satisfying $t(rare) = t(area)$.

Is some of it (or all of it) still valid in infinite algebras?

Pawel Idziak and Ralph McKenzie

Finitely decidable varieties

Following R. McKenzie and M. Valeriote's characterization of decidable varieties, many people contributed to an ongoing attempt to characterize finitely decidable varieties. Particularly, they sought a (mostly) structural characterization of those finite algebras **A** such that the variety $\mathcal{V}$ generated by **A** has decidable theory of its finite members. The problem is unsolved today, but the literature of the 1990's contains many contributions toward a possible solution, notably papers of Joohee Jeong, Pawel Idziak, Matthew Valeriote, Ross Willard and Dejan Delic.

In case **A** is finite and $\mathcal{V} = V(\mathbf{A})$ is finitely decidable and omits Hobby-McKenzie type **1**, Idziak showed that $\mathcal{V}$ has permuting congruences, has a finite bound on the size of its subdirectly irreducible members, and for every subdirectly irreducible algebra $\mathbf{S} \in \mathcal{V}$, the centralizer ζ of the monolith of **S** is comparable to all congruences, is the largest solvable congruence (the solvable radical), is Abelian, and congruences above ζ form a chain of type **3** covers. In fact, $\mathcal{V}$ has only type **2** and **3** congruence covers and the McKenzie-Valeriote $(3, 2)$-transfer property holds in $\mathcal{V}$. These structural properties of algebras, plus the condition that a certain ring derived from Abelian congruence covers in $\mathcal{V}$ has finitely decidable theory of modules, were shown by Idziak to characterize the finitely generated, finitely decidable, varieties that omit type **1**. So Idziak solved the characterization problem for these varieties.

The results obtained in the 1990's for the general case (where type **1** is allowed) showed that Idziak's properties still hold for finite SI's with monolith of type other than **1**. Around 2002, McKenzie showed (unpublished manuscript) that for a finite SI with monolith of type **1** in a finitely decidable variety, the centralizer ζ of the monolith is comparable to all congruences, is the solvable radical, is the largest Abelian congruence, and the congruences above ζ form a chain of type **3** covers. In fact, $\mathcal{V}$ has only type **1**, **2**, and **3** congruence covers and the $(3, 2), (3, 1)$ and $(2, 1)$ transfer properties hold in $\mathcal{V}$.

In the paper *Strong solvability and residual finiteness for finitely decidable varieties* (still being rewritten), Ralph McKenzie and Matthew Smedberg present these old results of McKenzie, and several new results from Smedberg's Vanderbilt PhD thesis. In particular, every finitely generated, finitely decidable variety has a finite residual bound. So there is a second unsolved problem obviously arising from this strand of investigation of finitely decidable varieties. We now state both problems:

(1) Find and prove a characterization of finitely decidable varieties along the lines of Idziak's characterization for the case where type **1** is omitted.
(2) Prove or disprove that every finite algebra **A** of finite signature that generates a finitely decidable variety, is finitely based (i.e., $\mathcal{V} = V(\mathbf{A})$ is finitely axiomatizable).

Problem (2) seems especially tempting. As far as I know, the probably is virtually unknown and nothing has been published on it. Progress on this problem has the potential to add to what we know on Problem (1), and either to solve the Jónsson-Park finite basis conjecture (i.e., that every finite algebra of finite signature in a variety with a finite residual bound, does possess a finite equational basis), or at least to inspire new approaches to that conjecture. Moreover, the algebras in question lie in varieties where all finite algebras have very strong structural properties.

Ralph McKenzie and Matthew Moore

Cube terms, Taylor terms, and the SD(meet)-condition

The following ten Maltsev conditions have undeniably been the most important Maltsev conditions in universal algebraic research. Maltsev term, cube-term, congruence modular, congruence distributive, Taylor term, and these congruence properties: SD($\wedge$), SD($\vee$), n-permutability, having a congruence equation, n-permutability plus SD($\vee$). "Having a Maltsev-term" is the strongest form of "having a cube-term". The final five conditions, plus "having a Taylor-term" were shown in Hobby-McKenzie to be equivalent to the six "omitting I" conditions, where I is any proper order-ideal in the five-element lattice of types of congruence covers. It is known that among the six omitting-types conditions listed, just "having a Taylor-term" and SD($\wedge$ (and no others) are equivalent to a strong Maltsev condition over the realm of all locally finite varieties. Miroslave Olšá, *The weakest nontrivial idempotent equations* (manuscript) proved this summer the remarkable result that "having a Taylor-term" is actually a strong Maltsev condition over the realm of all varieties. In light of the Valeriote paper, and other known results, the following open problem is the only one of its kind still open for any of the ten listed Maltsev properties.

Problem. Is congruence SD($\wedge$) equivalent to a strong Maltsev condition over the realm of all varieties? We have no idea how to attack this problem.

In a (rather unrelated) direction, we have some new results on cube-terms. First, Moore proved that any variety $\mathcal{V}$ has a cube-term iff the free algebra $\mathbf{F}_{\mathcal{V}}(x,y)$ cannot be "colored" in the graph $G = \langle\{0,1\}, R_n\ (n \geq 2)\rangle$, where

$$R_n = \{0,1\}^n \setminus \{(1,\dots,1)\}\rangle\,.$$

This graph is essentially the cube-term blocker $(\{0\},\{0,1\})$ which defines the clone of term operations of the two-element implication algebra. Then, constructed an example of a denumerable idempotent groupoid $\mathbf{B}$ (algebra with one binary operation) such that every non-trivial subalgebra of $\mathbf{B}$ generates the variety of all idempotent groupoids, so $\mathcal{V} = V(\mathbf{B})$ has no cube-term; and yet every two-element subset of $\mathbf{B}$ supports two term-derived operations that make it into a two-element lattice—so that $\mathbf{B}$ certainly has no cube-term blocker. McKenzie subsequently proved that any idempotent variety $\mathcal{V}$ has a cube-term iff none of the non-trivial free algebras in $\mathcal{V}$ (i.e., with an at least two-element set of free generators) has a cube-term blocker, iff $\mathbf{F}_{\mathcal{V}}(x,y)$ has no cube-term blocker. All three results are in the paper R. McKenzie and M. Moore, *Coloring and Blockers*, which is in preparation.

Cliff Bergman

Maltsev Products of Varieties and Quasivarieties

Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties, or even quasivarieties, of the same similarity type. Generalizing a construction of H. Neumann's for groups, Maltsev defined

$$\mathcal{V}\circ\mathcal{W} = \{\,\mathbf{A} : (\exists\theta\in\mathrm{Con}(\mathbf{A}))\ \mathbf{A}/\theta\in\mathcal{W}\ \&\ (\forall a\in A)\ a/\theta\in\mathrm{Sub}(\mathbf{A}) \implies a/\theta\in\mathcal{V}\,\}.$$

$\mathcal{V}\circ\mathcal{W}$ is now usually called the *Maltsev product of* $\mathcal{V}$ *and* $\mathcal{W}$. Actually, assuming $\mathcal{V}$ and $\mathcal{W}$ are subclasses of some larger quasivariety $\mathcal{U}$, Maltsev defined $\mathcal{V}\circ_{\mathcal{U}}\mathcal{W} = (\mathcal{V}\circ\mathcal{W})\cap\mathcal{U}$, which we shall call the Maltsev product *relative to* $\mathcal{U}$. With $\mathcal{U}$ taken to be the variety of all groups, the relative Maltsev product coincides with the product defined by Neumann.

If $\mathcal{V}$ and $\mathcal{W}$ consist of idempotent algebras, then the condition $a/\theta\in\mathrm{Sub}(\mathbf{A})$ will hold universally. In that case, $\mathcal{V}\circ\mathcal{W}$ will also be idempotent. In general, $\mathcal{V}\circ\mathcal{W}$ is not a variety. If the similarity type is finite, or if $\mathcal{W}$ is idempotent, it will always be a quasivariety.

Neumann, in her 1967 monograph *Varieties of Groups,* derives some strong and striking results for "her" product. By contrast, almost nothing seems to be known outside the group case. Freese and McKenzie have recently shown that a number of important Maltsev properties for idempotent varieties are preserved by the general product. It is not hard to prove that if $\mathcal{V}\circ_{\mathcal{U}}\mathcal{W}$ is congruence-permutable, then it is a variety (i.e., closed under H.)

The Maltsev product seems to be an important tool in the algebraic approach to the constraint satisfaction problem. We list here a few questions that it may be worthwhile to investigate.

(1) From axiomatizations of $\mathcal{V}$ and $\mathcal{W}$, can we deduce an axiomatization of $\mathcal{V}\circ\mathcal{W}$ or of $H(\mathcal{V}\circ\mathcal{W})$?

(2) Is $\mathcal{V} \circ \mathcal{W}$ ever finitely based?
(3) If $\mathcal{V}$ and $\mathcal{W}$ are locally finite, is $\mathcal{V} \circ \mathcal{W}$ locally finite? (I believe the answer is 'no', but perhaps for a well-chosen relativization it may sometimes hold.)
(4) Can we say anything about the free algebras in $\mathcal{V} \circ \mathcal{W}$?

As a starting point, one might consider a very simple case, such as taking $\mathcal{V}$ and $\mathcal{W}$ to be either semilattices or left-zero semigroups, and $\mathcal{U}$ to be semigroups or commutative binars.

Alexandr Kazda
Finding Ramsey

Having twice used the Ramsey theorem in universal algebraic situations, I'm wondering if there are more places in the study of relational structures where this theorem can be useful. Loosely speaking, Ramsey theorem guarantees us that large structures have nice substructures – and in a lot of situations in universal algebra what we want is to show that if there exists a counterexample relation of large arity then there exists a nice relation that is also a counterexample. While I don't think that Ramsey theorem is a silver bullet for every problem, it is a tool that I think can be used more.

As a specific project, I will outline a possible strategy for a proof that CSP of relational structures that generate join semidistributive varieties is solvable by linear Datalog (and thus belongs into the class NL).

Matthew Moore
Decidability in Universal Algebra

Many properties associated with the relational and operational clones of algebras are suspected to be undecidable. We will examine two such properties: finite relatedness and finite dualizability. In the spirit of the workshop, we also present some partial progress which establishes the undecidability of strictly weaker properties. The method in which these partial results are obtained may point towards a general proof.

George F. McNulty
Problems about finite algebras in varieites

This talk will set the context for the following open problems.

Problem 1. Let $\mathbf{A}$ be a finite algebra of finite signature and let $\mathcal{V}$ be the variety it generates. Consider the following properties $\mathcal{V}$ might have:
(a) $\mathcal{V}$ has a finite residual bound.
(b) There is a finitely based variety $\mathcal{W}$ so that $\mathcal{V} \subseteq \mathcal{W}$, $\mathcal{V}_{\text{fin}} = \mathcal{W}_{\text{fin}}$, and $\mathcal{W}$ is finitely based.
(c) $\mathbf{A}$ is dualizable.
(d) $\mathcal{V}$ has a Taylor term.

Which combinations of the above properties entail that $\mathcal{V}$ is finitely based?

Problem 2. Let $\mathbf{G}$ be Roger Bryant's pointed group and $\mathcal{V}$ be the variety generated by $\mathbf{G}$. What is the computational complexity of the Finite Algebra Membership Problem for $\mathcal{V}$? What is the equational complexity of $\mathcal{V}$?

Problem 3. Is there an algorithm, which, upon input of a finite algebra $\mathbf{A}$ of finite signature, will determine whether the Finite Algebra Membership Problem of $\mathsf{HSP}\mathbf{A}$ can be solved in polynomial time?

Petar Marković
Techniques and problems related to Quantified Constraint Satisfaction

I will present the current state of research on the Quantified Constraint Satisfaction Problem. Today we know that many analogues of the usual algebraic techniques for CSP can be found for QCSP, though some do not work quite as well. The polymorphisms are replaced by surjective polymorphisms, and the necessary and sufficient condition for $QCSP(\mathcal{A}) \subseteq QCSP(\mathcal{B})$ is given as existence of a surjective polymorphism from some finite power of $\mathcal{A}$ onto $\mathcal{B}$ (with a known bound on the power $|A|^{|B|}$). Also there is a workable definition of cores, though not as useful as in CSP (for starters, they are not unique up to isomorphism).

The situation is much better when the polymorphisms are all idempotent, i.e. when one assumes that all one-element unary relations are among the fundamental relations of the template. There a dichotomy was recently proved by Martin, related to the algebra of polymorphisms having polynomially (in NP) or exponentially generated powers (coNP-hard). Those are the only two possbilities, as we know from Zhuk's last year's result. A most important challenge is therefore to characterize the finite relational structures whose algebra of idempotent polymorphisms has PGP/EGP.

A well-known Pspace-complete complexity-forcing sufficient condition (for core templates) is that the algebra of idempotent polymorphisms is trivial. This forces the reduction from $QCSP(\mathcal{K}_3)$ to QCSP of the template. We will present conditions which force that the template has only idempotent polymorphisms. Again, there is a challenge to classify the relational structures which have only trivial idempotent polymorphisms.

Dmitriy Zhuk
The proof of CSP dichotomy conjecture
At the AAA conferences I presented an algorithm that solves the Constraint Satisfaction Problem. The main idea of the algorithm was to reduce the universes step by step to an absorbing set, or to a center, or to a solution of a system of linear equations. It can be proved that these reductions preserve 1-consistency. The only problem of the algorithm was that it probably doesn't work if we apply the linear reduction twice. Nevertheless, it can be proved that the algorithm solves CSP if the size of the universe is less than 8. At the same time, the algorithm probably doesn't work even for Mal'tsev case.

In my talk I will present a simpler algorithm that avoids such a gap. The main difference between the algorithms is that the new one always chooses an appropriate solution of a system of linear equations. First, I will explain the idea of the new algorithm on a simple Mal'tsev case. Then, I will describe it in general case. Finally, I will try to prove that this algorithm works in polynomial time and, therefore, the CSP dichotomy conjecture holds.

Andrew Moorehead
The structure of supernilpotent algebras in congruence modular varieties

An algebra belonging to a modular variety is abelian if and only if the difference term satisfies the mal'cev identities and is a homomorphism. This implies that the abelian algebras in congruence modular varieties are exactly the affine algebras. We will show that this is the k=1 case for more general classification of k-step supernilpotent algebras belonging to a modular variety. To do this, we generalize the notion of the delta congruence and strong cube terms introduced by Oprsal. The details have been worked out for the 3-ary commutator and a notation extending Oprsal's is being developed for higher arity. This leads us to a general question: how far can commutator theory be extended to higher commutator theory to achieve finer classification results?

Libor Barto
Loop Lemmata

A loop lemma is a theorem of the following form.

Let R be subdirect subset of A^2 (regarded as a digraph). Assume

- ...finiteness assumption..., and
- ...structural assumption..., and
- ...algebraic assumption....

Then R has a loop (ie. $(a, a) \in R$ for some $a \in A$)

Finiteness assumption can be eg. (F1) A finite, (F2) A countable and R compatible with an oligomorphic permutation group on A, or (F3) no finiteness assumption.

Structural assumptions are eg. (S1) R symmetric, contains a 3–cycle, (S2) R symmetric, contains an odd cycle, (S3) R is linked, (S4) R is strongly connected of algebraic length 1, (S5) R has algebraic length 1.

Algebraic assumptions are eg. (A1) R is compatible with an NU, (A2) R is compatible with an idempotent Taylor operation, (A3) R absorbs A^2 wrt. an idempotent operation, (A4) R absorbs the diagonal wrt. an idempotent operation.

A loop lemma by Barto, Kozik, Niven shows that (F1),(S5),(A2) or (A4) are sufficient. Several problem areas (motivated by curiosity, finite domain CSP, infinite domain CSP, and UA):

- **Purely infinite versions**. What combinations of (Sx) and (Ax) are sufficient? Olšák has shown that (S2), (A1) or (A4) are sufficient (this is a step towards the double loop lemma, which gives the weakest nontrivial idempotent condition). Kazda has shown that (S1), (A2) are not sufficient. The "easiest" open problem is (S3) or (S4), (A3).
- **Local versions**. Can the algebraic assumptions be further weakened to local assumptions? I have observed that if (F1), (S1), and R is compatible with f such that f is idempotent and $f(a, a, ..a, b, a..., a) = a$ for some a in a 3-cycle and some neigbor b, then R has a loop. The same is true without (F1) as observed by Olšák. The next step is to prove the finite version for 5–cycle. Another important special case seems to be a local version of (A4) $-$ R absorbs some loop plus, eg. (S1).
- **Pseudo versions**. With Pinsker we have proved that (F2), (S1), (A5) are sufficient to obtain a pseudoloop: a pair (a, b) within an orbit of the group action from (F2). The assumption (A5) is that the polymorphism clone of $(A; R, \text{orbits of } n\text{-tuples}, \text{finitely many constants})$ does not have a continuous homomoprhism to the clone of projections. Can (S1) be replaced by (S2)? Or even (S5)? There may be meaningful versions of pseudo-loop lemma even for finite A's, using pseudo-versions of structural assumptions.
- **Common loops**. With Kozik and Willard we have proved that if $R_1, \ldots, R_n$ are subdirect in A^2 and absorb the diagonal wrt. the same idempotent operation, then they have a common loop. Is there an infinite version, eg. is the following true? If $R_1, \ldots, R_n$ are subdirect and absorb A^2 wrt. the same idempotent operation, then they have a common loop.
- **Higher arity relations**. The double loop lemma of Olšák says that if R is a 4-ary subpower of an idempotent Taylor algebra generated by the twelve quadruples $(a_1, a_2, b_1, b_2) \in \{x, y\}^4$ with $a_1 \neq a_2$ or $b_1 \neq b_2$, then R contains a quadruple (c, c, d, d). The double loop lemma and some extra work gives that a ternary subpower of an idempotent Taylor algebra generated by the six non-constant triples $(a, b, c) \in \{x, y\}^6$ contains a constant triple. How to prove the latter theorem more directly? Is it enough to use only 4 generators?
- **Beyond the loops**. The conclusion of the loop lemma is that the intersection of A and the diagonal relation is non-empty, or that the CSP instance $R(x, y) \wedge (x = y)$ has a solution. Are there meaningful generalizations of the various pseudo/local/infinite loop lemmata to CSP instances?

A problem somewhat related to the local loop lemmata: Assume that $R \subseteq A^2$ is symmetric and connected, $B \subseteq A$ such that $R \cap (B \times B)$ is subdirect, and B absorbs A. Is there necessarily a homomorphism (or even retraction) $(A; R) \to (B; R \cap (B \times B))$?

Somewhat related are also problems such as the following. Assume $\mathbf{A}$ is a closed idempotent clone such that each finite subset $X \subseteq A$ has a 3–WNU (meaning that some f_X from the clone satisfies $f_X(x, x, y) = f_X(x, y, x) = f_X(y, x, x)$ for all $x, y \in X$) . Does $\mathbf{A}$ necessarily have a Taylor term.