FINITE BASE PROBLEMS DISCUSSION

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1. Introduction

First of all, is it "finite base" or "finite basis"? Never mind...

2. β function definitions and problems

In his talk, George McNulty gave the following definition of the beta function:

Definition 1. Let $\mathcal{V}$ be a variety in a finite signature, and n a positive integer. $\beta_{\mathcal{V}}(n)$ is the smallest positive integer ℓ such that any (finite) algebra $\mathbf{B}$ of cardinality less than n belongs to $\mathcal{V}$ iff for all pairs (s,t) of terms in the signature of $\mathcal{V}$ with $|s|+|t|<\ell$, if $\mathcal{V}\models s\approx t$ then $\mathbf{B}\models s\approx t$. ($|t|$ is the syntactic length of t.) The function $\beta_{\mathcal{V}}$ is called the *equational complexity* of $\mathcal{V}$.

Ross doesn't like this definition because (in principle at least; he can't think of an example off of the top of his head) $\beta_{\mathcal{V}}$ depends too much on the signature, that is, on the particular presentation of the clone of $\mathcal{V}$. (For example, Ross sees no reason why there there cannot exist two term-equivalent varieties $\mathcal{V},\mathcal{W}$ in finite signatures, with $\beta_{\mathcal{V}}$ bounded by a polynomial function while $\beta_{\mathcal{W}}$ is exponential.)

The CORRECT definition involves a generalization of terms called *circuits*. Given a finite signature, a *circuit* is a finite directed acyclic graph $(V,\to)$, that is, a digraph with no directed cycles, which is connected, has a unique sink vertex $\top$, and whose vertices and edges have the following kind of labelling:

(1) Each non-leaf vertex is labelled by a basic operation symbol of arity $n>0$.
(2) Each leaf is labelled by a variable or a constant symbol.
(3) For each non-leaf vertex v, if the label of v has arity n, then the edges coming into v are labelled with the integers $1,2,\dots,n$. (An edge can have multiple labels, but each $i=1,\dots,n$ labels exactly one edge coming into v.)

The term tree of a term is an example of a circuit. Conversely, every circuit recursively defines a term in the obvious way. The *size* of a circuit is its number of nodes. The advantage of circuits is that some terms can be represented by significantly smaller circuits.

Definition 2. Let $\mathcal{V}$ be a variety in a finite language, and n a positive integer. $\beta^*_{\mathcal{V}}(n)$ is defined just like $\beta_{\mathcal{V}}(n)$ except that in place of pairs of terms (s,t) one uses pairs of circuits.

It is easy to show that, given $\mathcal{V}$, β_V is bounded by a constant if and only if $\beta^*_{\mathcal{V}}$ is bounded by a constant. (So George and Ross can be friends.)

Problem 3. Does there exist a finite algebra $\mathbf{A}$ in a finite signature with the properties that $\mathbf{A}$ is not finitely based, but $\beta_{\mathcal{V}(\mathbf{A})}$ is bounded by a constant?

3. Finite base questions

A well-known, and now surpassed, theorem about finite bases is

Theorem 4 (Willard, 2000, [6]). *Let $\mathcal{V}$ be a locally finite, residually finite, congruence meet-semidistributive variety. Then $\mathcal{V}$ is finitely based.*

Definition 5. The tournaments are commutative idempotent groupoids in which for all x and y, the product $xy \in \{x, y\}$ (i.e. they are conservative).

One of the early applications of Willard's finite basis theorem was:

Theorem 6 (Miklós Mároti, 2002, [5]). *Every finite tournament has a finite base of equations.*

About this theorem: If one considers the subtraces, it is trivial to deduce that any tournament generates a congruence meet-semidistributive variety. In order to apply Willard's finite basis theorem, one needs to prove residual finiteness. First, it $\mathbf{T}$ is a finite tournament, one can limit the size of finite subdirectly irreducible tournaments in the variety $\mathcal{V}(\mathbf{T})$ to $2|T|$, next (and by far the hardest), prove that all subdirectly irreducible algebras in the variety are tournaments.
A natural generalization would be the following:

Definition 7. A 2-semilattice is an idempotent, commutative groupoid which satisfies the identity $x(xy) \approx xy$.

Clearly all tournaments are 2-semilattices and the same proof which proves that tournaments generate congruence meet-semidistributive varieties works also in 2-semilattices. So the other requirement is the issue:

Problem 8. Does every finite 2-semilattice generate a residually finite variety?

This problem might be very difficult if the answer is yes, since, as we said, the proof is very difficult even for tournaments. On the other hand, there may exist a different proof which is easier, or a counterexample 2-semilattice.
Another issue concerns a slate of problems, or conjectures posed by, or attributed to, Jonsson at the 1976 Oberwolfach meeting mentioned by George McNulty in his talk (see the summary in [2]). They were (always we assume a finite language, otherwise talking of finite equational base makes little sense):

Conjecture 1. If $\mathbf{A}$ is a finite algebra and the variety $\mathcal{V}$ it generates satisfies that $\mathcal{V}_{SI} \subseteq \mathsf{HS}(\mathbf{A})$, then $\mathcal{V}$ is finitely based.

This is easily seen to be equivalent to the more popular statement (independently posed by R. Park in his dissertation which came out that same year 1976):

Conjecture 2 (Jónsson-Park's conjecture). If $\mathcal{V}$ has a finite residual bound, then $\mathcal{V}$ is finitely based.

Another version attributed to Jónsson by R. McKenzie was refuted by McKenzie in mid-1990s:

Conjecture 3. If $\mathcal{V}$ is finitely generated and residually small (= has a cardinal residual bound), then $\mathcal{V}$ is finitely based.

Finally, the only version of the finite basis conjecture which actually made the report from that Oberwolfach meeting was:

Conjecture 4. If $\mathcal{V}_{SI}$ is a strictly elementary class, then $\mathcal{V}$ is finitely based.

These are all the conjectures of Jónsson from the 1976 Oberwolfach meeting about finite basis. However, Conjecture 2 can be plausibly strengthened a little, and the strengthening would be vacuous if the following Restricted Quackenbush Conjecture was proved:

Conjecture 5 (The Restricted Quachenbush Conjecture)**.** If $\mathcal{V}$ is finitely generated and contains no infinite subdirectly irreducible algebras, then $\mathcal{V}$ has a finite residual bound.

In other words, the restricted Quackenbush Conjecture states that a finitely generated variety in finite language can not have residual bound equal to $\aleph_0$. Thus the slightly stronger version of the Jónsson-Park Conjecture is

Conjecture 6. If $\mathcal{V}$ is finitely generated and residually finite, then $\mathcal{V}$ is finitely based.

The current state of the on the most famous two versions, Conjectures 2 and 6 are well-known: The first one is verified if $\mathcal{V}$ has a weak difference term, while the second one is verified whenever Conjecture 5 is, namely in congruence meet-semidistributive varieties and the varieties which satisfy a nontrivial congruence equation (see [3] and [4]). But a casual observer (like P. Marković) could mistakenly believe that Conjecture 4 was proved in the congruence distributive setting. However, this is not true, Jónsson in [1] actually proved a slightly different result:

Theorem 9. *If $\mathcal{V}_{FSI}$ is a strictly elementary class, then $\mathcal{V}$ is finitely based.*

This leaves the door open to two problems. Firstly,

Problem 10. If $\mathcal{V}$ is congruence distributive and $\mathcal{V}_{SI}$ is a strictly elementary class, must $\mathcal{V}$ be finitely based?

Also, it seems natural to consider the improved version of Conjecture 4, instead of Jónsson's original formulation:

Conjecture 7. If $\mathcal{V}_{FSI}$ is a strictly elementary class, then $\mathcal{V}$ is finitely based.

This Conjecture 7 might be considered in restricted settings, like algebras with a meet operation (expansions of a meet semilattice) or congruence join-semidistributive varieties.

About the possible tools for Conjecture 4, we only know what was known to Jónsson, namely his meta-theorem

Theorem 11 (Jónsson, see [1])**.** *Let $\mathcal{V}$ be a variety and $\mathcal{B}$ a strictly elementary class such that $\mathcal{V} \subseteq \mathcal{B}$. If there exists an elementary class $\mathcal{C}$ such that $\mathcal{B}_{SI} \subseteq \mathcal{C}$ and that $\mathcal{V} \cap \mathcal{C}$ is strictly elementary, then $\mathcal{V}$ has a finite base.*

and the theory of congruence distributive varieties (notably Jónsson's Lemma). We may also be able to use the following easily proved proposition:

Proposition 12. *An elementary class $\mathcal{A}$ is strictly elementary relative to the elementary class $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$ iff the complement $\mathcal{B} \setminus \mathcal{A}$ is closed under ultraproducts.*

For Conjecture 7 we may use the proof of Theorem 9 as a blueprint, namely we may strive to describe the property that $\mathrm{Cg}^{\mathbf{A}}(a,b) \cap \mathrm{Cg}^{\mathbf{A}}(c,d) \neq \emptyset$ by limiting the complexity of polynomials used in the Mal'cev chains and the lengths of those chains. For details on these approaches one may read more in the survey [7], which may be dated for questions like Conjectures 2, 5 and 6, but is still up to date on these other problems.

References

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[3] Kearnes, K. A., Willard, R., Residually finite, congruence meet-semidistributive varieties of finite type have a finite residual bound, Proc. Amer. Math. Soc. **127** no. 10 (1999), 2841–2850.

[4] Kearnes, K. A., Willard, R., Residually finite varieties, (in preparation, title not final).

[5] M. Maróti, The variety generated by tournaments, PhD thesis, Vanderbilt University, 2002

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[7] Willard, R., Extending Baker's theorem, Algebra Universalis **45** no. 2-3 (2001), 335–344.