# MAL'CEV PRODUCTS

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## 1. Introduction

The earliest mention of Mal'cev products of varieties were in a 1962 paper by Hanna Neumann, with ideas dating back to an earlier paper of hers from 1956. In her 1967 book *Varieties of groups* she presented the same material nicely. A. I. Mal'cev extended those ideas in 1967 (taken from an earlier lecture of his at the International Congress of Mathematicians in Moscow). Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be quasivarieties throughout the lecture.

**Definition 1.** $\mathcal{A} \circ \mathcal{B} = \{\mathbf{G} : (\exists\theta \in \operatorname{Con} \mathbf{G})\ (\mathbf{G}/\theta \in \mathcal{B} \,\&(\forall x \in G)(x/\theta \in \operatorname{Sub} \mathbf{G} \Rightarrow x/\theta \in \mathcal{A}))\}$.

We will write $[x]_\theta = \{y \in G : (x, y) \in \theta\}$ when we think of the $\theta$-class of $x$ as a set, and $x/\theta$ when we think of it as a member of the factor algebra $\mathbf{G}/\theta$ (though it is the same object, formally).

**Definition 2.** If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$, $\mathcal{A} \circ_\mathcal{C} \mathcal{B} = (\mathcal{A} \circ \mathcal{B}) \cap \mathcal{C}$.

Observe that $[a]_\theta \in \operatorname{Sub} \mathbf{G}$ iff for all basic operations $f$, $f(a, a, \ldots, a)\ \theta\ a$.

**Lemma 3.** *Let* $\mathbf{G}$ *be an algebra and* $\mathcal{A}$ *and* $\mathcal{B}$ *quasivarieties. Denote by* $\Lambda := \{\theta \in \operatorname{Con} \mathbf{G} : \mathbf{G}/\theta \in \mathcal{B}\}$ *and let* $\lambda = \bigcap \Lambda$. *Then* $\mathbf{G}/\lambda \in \mathcal{B}$ *and* $\mathbf{G} \in \mathcal{A} \circ \mathcal{B}$ *iff for all* $x \in G$, *if* $x/\lambda \in \operatorname{Sub} \mathbf{G}$, *then* $x/\lambda \in \mathcal{A}$.

*Proof.* By the subdirect decomposition we have $\mathbf{G}/\lambda \leq_{sd} \prod\limits_{\theta\in\Lambda} \mathbf{G}/\theta$. Since $\mathcal{B}$ is a quasivariety, it is closed under subdirect products, so $\mathbf{G}/\lambda \in \mathcal{B}$ and therefore $\lambda \in \Lambda$.

Since $\mathbf{G} \in \mathcal{A} \circ \mathcal{B}$, we select the congruence $\theta$ which satisfies the provisions of Definition 1. We wish to prove that $\lambda = \bigcap \Lambda$ also satisfies the same provisions. By the choice of $\theta$, $\mathbf{G}/\theta \in \mathcal{B}$, hence $\theta \in \Lambda$ by definition of $\Lambda$ and so we get $\lambda \leq \theta$. Assuming $[x]_\lambda \in \mathcal{A}$, then for all fundamental operations $f$, $f(a, a, \ldots, a)\ \lambda\ a$ hence from $\lambda \subseteq \theta$ follows that for all fundamental operations $f$, $f(a, a, \ldots, a)\ \theta\ a$, and therefore $[a]_\theta \in \operatorname{Sub} \mathbf{G}$. From the choice of $\theta$ follows that $[a]_\theta \in \mathcal{A}$ and this, in turn, together with $[a]_\lambda \leq [a]_\theta$ implies that $[a]_\lambda \in \mathcal{A}$. □

We conclude that we may always use $\lambda$ instead of some random $\theta \in \operatorname{Con} \mathbf{G}$. We call $\lambda$ the *verbal congruence induced on* $\mathbf{G}$ *by* $\mathcal{B}$. Of course, it depends only on $\mathbf{G}$ and $\mathcal{B}$, not on $\mathcal{A}$.

**Theorem 4.** *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be quasivarieties of finite similarity type. Then* $\mathcal{A} \circ \mathcal{B}$ *is a quasivariety. If* $\mathcal{B}$ *is idempotent, then "finite similarity type" can be dropped.*

**Definition 5.** A *pole* of the quasivariety $\mathcal{C}$ is a unary term $c(x)$ such that

(1) $\mathcal{C} \models c(x) \approx c(y)$ (it is constant) and

(2) For all basic operations $c$, $\mathcal{C} \models f(c(x), c(x), \ldots, c(x)) \approx c(x)$ ($c$ is an idempotent element).

There are two interesting special cases:

a) When $\mathcal{A}$ and $\mathcal{B}$ are both idempotent.
b) When $\mathcal{A}$ and $\mathcal{B}$ are both "polarized" (have a pole).

Note also that if $\mathcal{A}$ and $\mathcal{B}$ are both idempotent, then $\mathcal{A} \circ \mathcal{B}$ is idempotent.

If $\mathcal{A}$ and $\mathcal{B}$ are both idempotent, then several Mal'cev conditions are preserved under $\circ$ (as proved in [1] by R. Freese and R. McKenzie):

- Having a Taylor term,
- having a cube term,
- congruence meet-semidistributivity,
- Having a near-unanimity term,
- Existence of some $n \in \omega$ such that a variety is congruence $n$-permutable.

**Theorem 6** (C. Bergman)**.** *If $\mathcal{A}$ and $\mathcal{B}$ are both idempotent, congruence permutable varieties such that $\mathcal{A} \vee \mathcal{B}$ is congruence permutable, then $\mathcal{A} \circ \mathcal{B}$ is congruence permutable.*

**Theorem 7** (A. I. Mal'cev)**.** *If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ are idempotent varieties and $\mathcal{C}$ is congruence permutable, then $\mathcal{A} \circ_{\mathcal{C}} \mathcal{B}$ is a variety.*

*Proof.* Let $\mathbf{R} \in \mathcal{A} \circ_{\mathcal{C}} \mathcal{B}$, $\alpha \in \operatorname{Con} \mathbf{R}$. We wish to prove $\mathbf{R}/\alpha \in \mathcal{A} \circ_{\mathcal{C}} \mathcal{B}$. Let $\lambda$ be the verbal congruence of $\mathbf{R}$, $\overline{\lambda} := \lambda \vee \alpha$. We know that $\overline{\lambda}/\lambda \in \operatorname{Con} \mathbf{R}/\lambda$ and $\overline{\lambda}/\alpha \in \operatorname{Con} \mathbf{R}/\alpha$. We get $(\mathbf{R}/\alpha)/(\overline{\lambda}/\alpha) \cong \mathbf{R}/\overline{\lambda} \cong (\mathbf{R}/\lambda)/(\overline{\lambda}/\lambda) \in \mathsf{H}(\mathbf{R}/\lambda) \subseteq \mathsf{H}(\mathcal{B}) = \mathcal{B}$. Moreover, let $A = [r]_\lambda \in \mathcal{A}$. Denote by $A^\alpha := \{x \in R : (\exists a \in A) x \; \alpha \; a\}$.

$$x \in [r]_{\overline{\lambda}} \iff (x, r) \in \overline{\lambda} = \alpha \vee \lambda = \lambda \circ \alpha \iff (\exists a \in A) \; x \; \alpha \; a \; \lambda \; r \iff x \in A^\alpha.$$

So we obtain $[r/\alpha]_{(\overline{\lambda}/\alpha)} = ([r]_{\overline{\lambda}})/\alpha = (\mathbf{A}^\alpha)/\alpha \cong \mathbf{A}/\alpha \in \mathcal{A}$. Since we know that $\mathcal{A} \circ_{\mathcal{C}} \mathcal{B}$ is closed under $\mathsf{S}$ and $\mathsf{P}$ (as it is a quasivariety), $\mathcal{A} \circ_{\mathcal{C}} \mathcal{B}$ is a variety. □

**Example 8.** Let $\mathcal{S}q$ be the variety of Steiner quasigroups (squags) with one binary operation. The equational base of $\mathcal{S}q$ is

$$xx \approx x$$
$$xy \approx yx$$
$$x(xy) \approx y$$

Define by $q(x, y, z) := y(xz)$, which is a Mal'cev term for $\mathcal{S}q$. Clearly, $\mathcal{S}q \vee \mathcal{S}q = \mathcal{S}q$, which is a congruence permutable variety, so by Theorem 6, $\mathcal{S}q \circ \mathcal{S}q$ is a variety.

Problems

**Problem 9.** Find an equational base for $\mathcal{A} \circ \mathcal{B}$. In particular, how about $\mathcal{S}q \circ \mathcal{S}q$?

**Problem 10.** More generally, if we have axiomatizations of $\mathcal{A}$ and $\mathcal{B}$, can we find an axiomatization of $\mathcal{A} \circ \mathcal{B}$ (or $\mathcal{A} \circ_{\mathcal{C}} \mathcal{B}$)?

**Problem 11.** If $\mathcal{A}$ and $\mathcal{B}$ are both locally finite, what about $\mathcal{A} \circ \mathcal{B}$? Ian Payne says: pretty much never. And $\mathcal{A} \circ_{\mathcal{C}} \mathcal{B}$?

**Problem 12.** What can we say about $\mathbf{F}_{\mathcal{A} \circ \mathcal{B}}(X)$?

For starters, $\mathbf{F}_{\mathcal{A}\circ\mathcal{B}}(X)/\lambda \cong \mathbf{F}_{\mathcal{B}}(X)$.

Let $\mathcal{S}emi$ be the variety of semilattices and $\mathcal{S}g$ the variety of semigroups. Define $\mathcal{Q} = \mathcal{S}emi \circ_{\mathcal{S}g} \mathcal{S}emi$. Then $\mathbf{F}_{\mathcal{Q}}(x, y)$ has the following $\lambda$-classes: $\{x\}$, $\{y\}$ and $\{$everything else such as $xy, yx, x(xy), ..\}$.

**Problem 13.** Axiomatize $\mathcal{S}emi \circ \mathcal{S}emi$. Or the quasivariety $\mathcal{Q}$ above.

## References

[1] Freese, R., McKenzie, R.: Maltsev families of varieties closed under join or Maltsev product, Algebra Universalis, (in press).

[2] A. I. Mal'cev, *Multiplication of classes of algebraic systems*, Siberian Math. J. **8** (1967), 254–267, translated in "The Metamathematics of Algebraic Systems. Collected Papers 1936–1967," North Holland, 1971.

[3] H. Neumann, *Varieties of groups*, Springer–Verlag, Berlin, 1967.