MAL'CEV CHARACTERIZATIONS OF CONGRUENCE MEET-SEMIDISTRIBUTIVITY IN LOCALLY FINITE VARIETIES

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These problems were posed on Tuesday, September 20th by Ralph McKenzie and Petar Marković.

The following result was proved by Marcin Kozik in 2009, appeared in the paper [3], using the strong version of the bounded width theorem by Libor Barto from [1]:

Theorem 1. *Let $\mathcal{V}$ be a locally finite variety. $\mathcal{V}$ is congruence meet-semidistributive iff there exist $\mathcal{V}$-terms $p(x, y, z)$ and $q(x, y, z, u)$ which are both weak near-unanimity terms in $\mathcal{V}$ and such that $\mathcal{V} \models p(x, x, y) \approx q(x, x, x, y)$.*

A proof which first copies Kozik's almost verbatim and then uses a compactness argument can be devised to prove (see [2]):

Theorem 2. *Let $\mathcal{V}$ be a locally finite variety. $\mathcal{V}$ is congruence meet-semidistributive iff for all $k \geq 3$ there exist $\mathcal{V}$-terms $p_k(x_1, \dots, x_k)$ which are weak near-unanimity terms in $\mathcal{V}$ and a $\mathcal{V}$-term $b(x, y)$ such that for all $k \geq 3$, $\mathcal{V} \models p_k(x, \dots, x, y) \approx b(x, y)$.*

Problem 3. Is Theorem 2 still true if we add "and $\mathcal{V} \models b(x, b(x, y)) \approx b(x, y)$" at the end of its statement (i.e. the requirement that all p_k are special in the terminology of [4])?

One partial result is known from [4]:

Theorem 4. *Let $\mathcal{V}$ be a locally finite variety. $\mathcal{V}$ is congruence meet-semidistributive iff there exists $n \geq 3$ such that for all $k \geq n$ there exist $\mathcal{V}$-terms $p_k(x_1, \dots, x_k)$ which are weak near-unanimity terms in $\mathcal{V}$ and a $\mathcal{V}$-term $b(x, y)$ such that for all $k \geq n$, $\mathcal{V} \models p_k(x, \dots, x, y) \approx b(x, y) \approx b(x, b(x, y))$.*

Another strong Mal'cev characterization of the congruence meet-semidistributivity in locally finite varieties proved in [2] is

Theorem 5. *Let $\mathcal{V}$ be a locally finite variety. $\mathcal{V}$ is congruence meet-semidistributive iff there exists a $\mathcal{V}$-term $p(x, y, z, u)$ such that $\mathcal{V} \models p(x, x, x, x) \approx x$ and $\mathcal{V} \models p(x, x, x, y) \approx p(x, x, y, x) \approx p(x, y, x, x) \approx p(y, x, x, x) \approx p(x, x, y, y) \approx p(x, y, x, y) \approx p(x, y, y, x)$.*

This strong Mal'cev condition is pretty strong syntactically and it implies most other known Mal'cev characterizations of congruence meet-semidistributivity in locally finite varieties. However, a computer search identified two other strong Mal'cev conditions which are even stronger than the one mentioned in Theorem 5, so they imply congruence meet-semidistributivity, but may be actually equivalent to it in all locally finite varieties. They are:

(BD)
$$\begin{aligned} t(x,x,x,x) &\approx x \\ t(x,x,y,z) &\approx t(y,z,y,x) \approx t(x,z,z,y) \end{aligned}$$

(LS)
$$\begin{aligned} t(x,x,x,x) &\approx x \\ t(x,x,y,z) &\approx t(y,x,z,x) \approx t(y,z,x,y) \end{aligned}$$

Problem 6. Does every locally finite congruence meet-semidistributive variety satisfy the strong Mal'cev condition (BD)? And how about (LS)?

Miklós Maróti solved Problem 6 on Friday, September 23rd by providing two counterexamples which are algebras with six elements. Both generate congruence meet-semidistributive varieties, but the first one fails the condition (BD) while the other one fails the condition (LS).

References

[1] Barto, L.: The collapse of the bounded width hierarchy. J. Logic Comput. **26** no. 3 (2016), 923–943.

[2] Jovanović, J., Marković, P., McKenzie, R., Moore, M.: Optimal strong Malcev conditions for congruence meet-semidistributivity in locally finite varieties, Algebra Universalis (in press), 21pp.

[3] Kozik, M., Krokhin, A., Valeriote, M. Willard, R.: Characterizations of several Maltsev conditions. Algebra Universalis **73** no. 3 (2015), 205-224,

[4] Maróti, M., McKenzie, R.: Existence theorems for weakly symmetric operations. Algebra Universalis **59** no. 3-4 (2008), 463–489