LOCAL MAL'CEV-LIKE CONDITIONS

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All operations which will be discussed are idempotent. By a *strong Mal'cev condition* we mean a finite set of identities in some language. Informally, a strong Mal'cev condition is *realized* in an algebra $\mathbf{A}$ (or variety $\mathcal{V}$) if there is a way to interpret the function symbols appearing in the condition as term operations of $\mathbf{A}$ (or $\mathcal{V}$) so that the identities in the Mal'cev condition become true equations in $\mathbf{A}$ (or $\mathcal{V}$). Another way of saying that an algebra $\mathbf{A}$ realizes a condition $\mathcal{C}$ is saying that $\mathbf{A}$ has a $<$description of the condition $\mathcal{C}>$-term.

Example 1. If an algebra has a commutative idempotent binary term operation $+$ then this algebra has a 4-ary weak near-unanimity term. Proof: just use $w(x, y, z, u) = (x + y) + (z + u)$.

Now we introduce the local satisfaction of a Mal'cev condition.

Definition 2. Let $\mathbf{A}$ be an algebra, let $\mathcal{C}$ be a strong Mal'cev condition and and let $X \subseteq A$. We say that $\mathbf{A}$ realizes $\mathcal{C}$ on X if there is a way to interpret the operations as terms of $\mathbf{A}$ so that for all evaluations of the variables as elements of X, all identities in $\mathcal{C}$ are true in those evaluations.

Problem 3. Let $\mathbf{A}$ be an algebra and $X \subseteq A$. If $\mathbf{A}$ has a commutative binary term on X, must $\mathbf{A}$ also have a 4-ary weak near-unanimity term on X?

<u>Motivations</u>:

(1) Continuous (for usual implications) vs. uniformly continuous (for local implications) clone homomorphisms.
(2) Local is simpler than global!

Definition 4. *The double loop term* is a strong Mal'cev condition in one 12-ary operation symbol t and two identities plus idempotence. It is given by

$$t\left(\begin{bmatrix}x\\x\\x\\y\end{bmatrix},\begin{bmatrix}x\\x\\y\\x\end{bmatrix},\begin{bmatrix}y\\y\\x\\y\end{bmatrix},\begin{bmatrix}y\\y\\y\\x\end{bmatrix},\begin{bmatrix}x\\y\\x\\x\end{bmatrix},\begin{bmatrix}y\\x\\x\\x\end{bmatrix},\begin{bmatrix}x\\y\\y\\y\end{bmatrix},\begin{bmatrix}y\\x\\y\\y\end{bmatrix},\begin{bmatrix}x\\y\\x\\y\end{bmatrix},\begin{bmatrix}x\\y\\y\\x\end{bmatrix},\begin{bmatrix}y\\x\\x\\y\end{bmatrix},\begin{bmatrix}y\\x\\y\\x\end{bmatrix}\right)=\begin{bmatrix}a\\a\\b\\b\end{bmatrix},$$

i.e. one identity between the terms in the first two rows and the other identity is between the terms in the second two rows of the above figure.

Problem 5. Let $\mathbf{A}$ be an algebra and $X \subseteq A$. If $\mathbf{A}$ has a Taylor term on X, must $\mathbf{A}$ also have a 12-ary double loop term on X?

We fix a variety $\mathcal{V}$ with a Taylor term t. Let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(X)$ and $X = \{x, y\}$.

Define $\mathbf{Q} \leq \mathbf{F}^4$ by

$$Q = \mathrm{Sg}^{\mathbf{A}^4}\left(\left\{\begin{bmatrix}x\\x\\x\\y\end{bmatrix},\begin{bmatrix}x\\x\\y\\x\end{bmatrix},\begin{bmatrix}y\\y\\x\\y\end{bmatrix},\begin{bmatrix}y\\y\\y\\x\end{bmatrix},\begin{bmatrix}x\\y\\x\\x\end{bmatrix},\begin{bmatrix}y\\x\\x\\x\end{bmatrix},\begin{bmatrix}x\\y\\y\\y\end{bmatrix},\begin{bmatrix}y\\x\\y\\y\end{bmatrix},\begin{bmatrix}x\\y\\x\\y\end{bmatrix},\begin{bmatrix}x\\y\\y\\x\end{bmatrix},\begin{bmatrix}y\\x\\x\\y\end{bmatrix},\begin{bmatrix}y\\x\\y\\x\end{bmatrix}\right\}\right)$$

and define $\mathbf{R} \leq \mathbf{F}^2$ by

$$\begin{bmatrix}a\\b\end{bmatrix} \in R \text{ iff } (\exists c \in F)\begin{bmatrix}a\\b\\c\\c\end{bmatrix} \in Q.$$

By the definition of Q we know

$$\begin{bmatrix}x\\y\end{bmatrix},\begin{bmatrix}y\\x\end{bmatrix} \in R$$

Moreover, since the set of all generators of Q is invariant under transposition of the first two coordinates, then so is the whole of Q. It follows that R is also invariant under the transposition of the first two coordinates as

$$\begin{bmatrix}a\\b\end{bmatrix} \in R \text{ iff } (\exists c \in F)\begin{bmatrix}a\\b\\c\\c\end{bmatrix} \in Q \text{ iff } (\exists c \in F)\begin{bmatrix}b\\a\\c\\c\end{bmatrix} \in Q \text{ iff } \begin{bmatrix}b\\a\end{bmatrix} \in R.$$

The above argument proves that R is a symmetric relation. Next we prove

Theorem 6. *The set* $A := \left\{\begin{bmatrix}x\\y\end{bmatrix},\begin{bmatrix}y\\x\end{bmatrix}\right\}$ *absorbs the set* $B := \left\{\begin{bmatrix}x\\x\end{bmatrix},\begin{bmatrix}y\\y\end{bmatrix}\right\}$ *into* R.

Proof. We will prove absorption via the Taylor term t.

Indeed, let $i \leq n$, let $[a_1, b_1]^T, \ldots, [a_{i-1}]^T, [a_{i+1}]^T, \ldots, [a_n, b_n]^T \in A$ and let $[a_i, b_i]^T \in B$ (of course, all $a_j, b_k \in \{x, y\}$). Since t is Taylor, there exist $c_1, \ldots, c_n$, $d_1, \ldots, d_n \in \{x, y\}$ such that $c_i = x$, $d_i = y$ and $t(c_1, \ldots, c_n) \approx t(d_1, \ldots, d_n)$. Let $t(c_1, \ldots, c_n) = z$. Note that for all $j \neq i$, $[a_j, b_j, c_j, d_j]^T \in Q$ (actually they are among the generators of Q) since $[a_j, b_j]^T \in A$, so $\{a_j, b_j\} = \{x, y\}$. Moreover, either $[a_i, b_i, c_i, d_i]^T = [x, x, x, y]^T$ or $[a_i, b_i, c_i, d_i]^T = [y, y, x, y]^T$, so $[a_i, b_i, c_i, d_i]^T \in Q$. We compute

$$t\left(\begin{bmatrix}a_1\\b_1\\c_1\\d_1\end{bmatrix},\begin{bmatrix}a_2\\b_2\\c_2\\d_2\end{bmatrix},\ldots,\begin{bmatrix}a_n\\b_n\\c_n\\d_n\end{bmatrix}\right) = \begin{bmatrix}u\\v\\z\\z\end{bmatrix}$$

So

$$t\left(\begin{bmatrix}a_1\\b_1\end{bmatrix},\begin{bmatrix}a_2\\b_2\end{bmatrix},\ldots,\begin{bmatrix}a_n\\b_n\end{bmatrix}\right) \in R, \text{ as desired.}$$

□

We would like to prove that in the above situation R has a loop, but we can't do it. So we have

Problem 7. Prove or disprove that R must have a loop.

We can prove Problem 7 when t is 3-ary:

Theorem 8. *Let* $A := \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix} \right\}$ *absorb the set* $B := \left\{ \begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix} \right\}$ *into* R *via a ternary idempotent operation* t*. Then* R *contains a loop.*

Proof. Notation: to avoid even longer expressions, we will write (abc) for $t(a, b, c)$. Denote $a = ((xyy)(xyx)(xxy))$, $b = ((yxx)(yxy)(xxy))$ and $c = ((yxx)(xyx)(yyx))$. We will force the loop at (abc). We do it by proving that $[(abc), a]^T \in R$, $[(abc), b]^T \in R$ and $[(abc), c]^T \in R$ and then applying t to those three vectors (and using idempotence). Indeed, $\begin{bmatrix} (abc) \\ a \end{bmatrix} =$

$$\left(\begin{matrix} ((xyy) & (xyx) & (xxy)) & ((yxx) & (yxy) & (xxy)) & ((yxx) & (xyx) & (yyx)) \\ (\;\; x & y & y \;\;) & ((xyx) & (xyx) & (xyx)) & ((xxy) & (xxy) & (xxy)) \end{matrix} \right).$$

Here the first three vector columns are in R since A absorbs B into R via t, while the remaining six are in t applied to vectors in A. Similarly, $\begin{bmatrix} (abc) \\ b \end{bmatrix} =$

$$\left(\begin{matrix} ((xyy) & (xyx) & (xxy)) & ((yxx) & (yxy) & (xxy)) & ((yxx) & (xyx) & (yyx)) \\ ((yxx) & (yxx) & (yxx)) & (\;\; y & x & y \;\;) & ((xxy) & (xxy) & (xxy)) \end{matrix} \right).$$

Here we apply absorption to the middle three vector columns. Finally, $\begin{bmatrix} (abc) \\ c \end{bmatrix} =$

$$\left(\begin{matrix} ((xyy) & (xyx) & (xxy)) & ((yxx) & (yxy) & (xxy)) & ((yxx) & (xyx) & (yyx)) \\ ((yxx) & (yxx) & (yxx)) & ((xyx) & (xyx) & (xyx)) & (\;\; y & y & x \;\;) \end{matrix} \right).$$

□

As a side note, Libor was able to prove it using t of arity 5, however he needs to to assume finiteness.

Note that R must contain an odd cycle. Indeed, let $a_1^0, \ldots, a_n^0 = x$ and for $0 < i \le n$ define $a_i^i = a_i^{i-1}$, while for $j \neq i$, if $a_j^{i-1} = x$ then $a_j^i = y$, while if $a_j^{i-1} = y$ then $a_j^i = x$. Now compute

$$\begin{bmatrix} b^0 \\ b^1 \\ \vdots \\ b^n \end{bmatrix} := t \left(\begin{bmatrix} a_1^0 \\ a_1^1 \\ \vdots \\ a_1^n \end{bmatrix}, \begin{bmatrix} a_2^0 \\ a_2^1 \\ \vdots \\ a_2^n \end{bmatrix}, \ldots, \begin{bmatrix} a_n^0 \\ a_n^1 \\ \vdots \\ a_n^n \end{bmatrix} \right)$$

For all $0 \le i < n$, from Theorem 6 follows that $[b^i, b^{i+1}]^T \in R$. Moreover, $b^0 = x$ by idempotence and if n is odd, then $b^n = t(x, x, \ldots, x) = x$ so we have an odd cycle in R, while if n is even, then $b^n = t(y, y, \ldots, y) = y$ and since $[y, x]^T \in R$, we get an odd cycle in R again.

Denote by $R^{\to k} := R \circ R \circ \ldots \circ R$ (using $k - 1$ symbols $\circ$). Since there is an odd cycle in R, and R is a symmetric relation, then R contains odd cycles of all except finitely many lengths. Assume that R has no loop. Let k be maximal such that $R^{\to 3^k}$ contains no loop. $R^{\to 3^k}$ contains a triangle. So, if we were able to prove that triangle implies loop, we would be in business for Problem 7.

By assuming an additional absorption condition, we can complete this proof. We will prove two statements by a simultaneous induction on the arity of the term n which is used in the absorption.

Theorem 9. *Let R be binary, symmetric, contains a triangle a, b, c and let a^+ absorb a. Then R has a loop.*

Theorem 10. *Let $\mathbf{A}$ be an algebra with a Taylor term t. Let R be a symmetric, nonempty binary relation on A, $R \lhd \Delta_A$. Then R contains a loop.*

Proof of the base case $n = 1$ of Theorem 9. The base case follows from idempotence. Indeed, $t(a) \in a^+$ by the fact that a^+ absorbs a via t. Therefore, $[a, a]^T \in R$ is the desired loop. □

Now come the inductive steps.

Proof of Theorem 9 assuming Theorem 10 for $n - 1$. : Let the arity n of t be greater than 1. Define $R' := R \cap a^+$ and $A' = \{x \in a^+ : x$ is not an isolated vertex in $R'\}$. We know that $b, c \in A'$ and $[b, c]^T \in R'$. We define a new "strong absorption" by $B \lll A$ if there exists a term s in more than one variable such that whenever at most two entries in a tuple $\overline{a}$ are in A, but the rest are in B, then $s(\overline{a}) \in B$. Finally, denote $S := R' \circ R' \circ R'$.

We claim that $S \lll (A')^2$. We will put the vectors in $(A')^2$ on the outside to be easier to see what we are doing but it should be obvious they can be on any two positions in the term. We assume that $[a_1, d_1]^T, [a_n, d_n]^T \in (A')^2$ and that for all $1 < i < n$, $[a_i, b_i]^T, [b_i, c_i]^T, [c_i, d_i]^T \in R'$. We select any c_1 and b_n such that $[c_1, d_1]^T, [a_n, b_n]^T \in R'$ (we can because d_1 and a_n are not isolated vertices).

$$t\left(\begin{bmatrix} a_1 \\ a \\ c_1 \\ d_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}, \dots, \begin{bmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \\ d_{n-1} \end{bmatrix}, \begin{bmatrix} a_n \\ b_n \\ a \\ d_n \end{bmatrix}\right) = \begin{bmatrix} a' \\ b' \\ c' \\ d' \end{bmatrix}$$

Note that $b', c' \in a^+$ since a^+ absorbs a and that none of a', b', c', d' are isolated. So we have for the term $t' = t(x_1, x_2, \dots, x_{n-1}, x_1)$ that A' and S satisfy that S absorbs $(A')^2$ via t'. S is clearly nonempty and symmetric. Therefore we have more than enough to apply Theorem 10 and deduce that the graph (A', S) contains a loop. This loop is a triangle a', b', c' in (A', R'). Together with a we get a complete R-graph $\mathbf{K}_4$ on the vertices $\{a, a', b', c'\}$.

So with respect to the graph (A', R') we have all the same assumptions as in the statement of Theorem 9. Note that since x^+ absorbs x with x^+ computed relative to R, then x^+ absorbs x with x^+ computed relative to R'. This means that we can continue and find a triangle in a'^+ (computed with respect to R') and therefore a complete graph with 5 vertices in (A, R). Continuing like this, we get as large a finite clique as we need in R, and we need the size $n + 1$. Let this clique be on the vertices $\{a, x_1, x_2, \dots, x_n\}$. Define $y_1 = t(a, x_1, \dots, x_1)$, $y_2 = t(x_2, a, x_2, \dots, x_2)$ and so on, till $y_n = t(x_n, x_n, \dots, x_n, a)$. Then we claim $[t(y_1, y_2, \dots, y_n), y_i]^T \in R$ for all $1 \le i \le n$. We show how this works for $i = 2$, you can imagine the analogous arguments for other i:

$$t\left(\begin{bmatrix} t(a, x_1, \dots, x_1) \\ x_2 \end{bmatrix}, \begin{bmatrix} t(x_2, a, x_2, \dots, x_2) \\ a \end{bmatrix}, \dots, \begin{bmatrix} t(x_n, x_n, \dots, x_n, a) \\ x_2 \end{bmatrix}\right) \in R,$$

using the fact that R absorbs the diagonal. Finally we prove that there is a loop at $t(y_1, y_2, \dots, y_n)$ using idempotence. □

Now we turn to Theorem 10.

Proof of Theorem 10 assuming Theorem 9 for n. First note that there exists an odd cycle in R. If we just take an element $c \in A$ which is not isolated in R and if $[c, d]^T \in R$, then the argument above the statement of Theorem 10 with c playing the role of x and d playing the role of y proves that R has an odd cycle. As noted before, since R is symmetric and contains an odd cycle, if there were no loops in R, then there exists the largest k such that the power $R^{\to 3^k}$ contains no loops. So, we may replace R by $R^{\to 3^k}$ and note that $R^{\to 3^k}$ still absorbs Δ_A, it is still symmetric and nonempty , but it also contains a 3-cycle. If we were to prove that $R^{\to 3^k}$ contained a loop, we would be done by contradiction. The letter S will from now on stand in place of $R^{\to 3^k}$.

Let $a \in F$ be such that a^+ (with respect to S) is nonempty. Then we claim that a^+ absorbs a. Indeed, let t be the term of arity n modulo which S absorbs Δ_A. Then for any $1 \leq i \leq n$ and any $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n \in a^+$,

$$t\left(\begin{bmatrix} a \\ b_1 \end{bmatrix}, \ldots, \begin{bmatrix} a \\ b_{i-1} \end{bmatrix}, \begin{bmatrix} a \\ a \end{bmatrix}, \begin{bmatrix} a \\ b_{i+1} \end{bmatrix}, \ldots, \begin{bmatrix} a_n \\ b_n \end{bmatrix}\right).$$

Hence, $[a, t(b_1, \ldots, b_{i-1}, a, b_{i+1}, \ldots, b_n)]^T \in S$, so $t(b_1, \ldots, b_{i-1}, a, b_{i+1}, \ldots, b_n) \in a^+$, proving the desired absorption.

Finally we apply Theorem 9 to S to derive that the graph (A, S) has a loop. $\square$