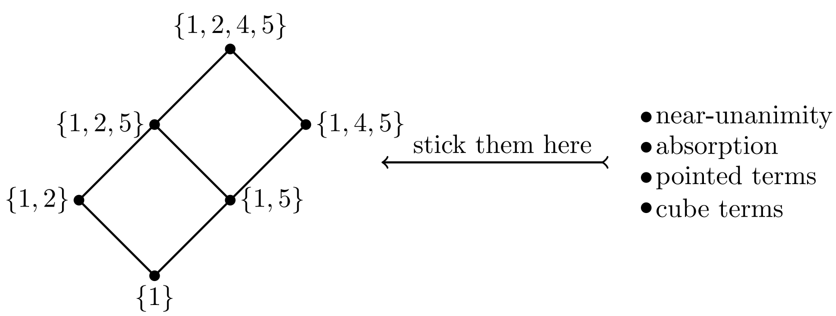
A VERY POINTED LECTURE

ALEX WIRES

1. Introduction

Some embarrassing story about P. Marković harassing graduate students ... the less said the better. Anyway, this story gave birth to the following ambition:



2. Pointed elements and terms

Definition 1. Let $\mathbf{A}$ be an algebra and $a \in A$. a is *pointed* if there exist a term $t(x_1, \ldots, x_n)$ and elements $a_1, \ldots, a_n \in A$ such that for all $1 \le i \le n$ and all $x \in A$, $t(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = a$.

Theorem 2 (M. Kozik, see [1])**.** *Let $\mathbf{A}$ be a finite idempotent algebra. Then $\mathcal{V}(\mathbf{A})$ is congruence meet-semidistributive iff every subalgebra $\mathbf{B} \le \mathbf{A}$ has a pointed element.*

The rest of this lecture is from the paper [2].

Definition 3. Let $\mathbf{A}$ be an algebra and $a \in A$. a is *cubed* if there exist a term $t(x_1, \ldots, x_n)$, elements $a_1, \ldots, a_n \in A$ and a cover $\mathcal{C}$ of $[n]$ consisting of proper subsets of $[n]$ such that a is pointed by t and for all $x \in A$ and $S \in \mathcal{C}$, $t(b_1, \ldots, b_n) = a$, where $b_i = x$ if $i \in S$, while $b_i = a_i$ if $i \notin S$.

Theorem 4. *Let $\mathbf{A}$ be a finite idempotent algebra. $\mathcal{V}(\mathbf{A})$ is Taylor iff every $\mathbf{B} \le \mathbf{A}$ has a cubed element.*

Definition 5. Let $\mathbf{A}$ be an algebra and $a \in A$. a is *strictly pointed* if there exist a term $t(x_1, \ldots, x_n)$, a cover $\mathcal{C}$ of $[n]$ consisting of proper non-singleton subsets of $[n]$ and elements $a_1, \ldots, a_n \in A$ such that for all $1 \le i \le n$ and all $x \in A$, $t(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = a$, while for all $S \in \mathcal{C}$, $t(b_1, \ldots, b_n) = x$, where $b_i = x$ if $i \in S$, while $b_i = a_i$ if $i \notin S$.

Notice: at the end of Definition 5, the result is x, not a like in Definition 3! An example of an algebra in which every element is a strictly pointed element is any algebra with the near-unanimity term.

Proposition 6. *Let $\mathbf{A}$ be a finite idempotent algebra. If every subalgebra $\mathbf{B} \leq \mathbf{A}$ has a strictly pointed element, then $\mathcal{V}(\mathbf{A})$ is congruence join-semidistributive.*

Definition 7. Let $\mathbf{A}$ be an algebra and $a \in A$. a is *strictly cubed* if there exist a term $t(x_1, \ldots, x_n)$, a cover $\mathcal{C}$ of $[n]$ consisting of proper subsets of $[n]$ and elements $a_1, \ldots, a_n \in A$ such that $t(a_1, \ldots, a_n) = a$, for all $1 \leq i < j \leq n$ and all $x \in A$, $t(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = x = t(a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_n)$ and finally, for all $S \in \mathcal{C}$, $t(b_1, \ldots, b_n) = a$, where $b_i = x$ if $i \in S$, while $b_i = a_i$ if $i \notin S$.

Proposition 8. *Let $\mathbf{A}$ be a finite idempotent algebra. If every subalgebra $\mathbf{B} \leq \mathbf{A}$ has a strictly pointed element, then $\mathcal{V}(\mathbf{A})$ is congruence n-permutable.*

Problem 9. Prove the other implication in Propositions 8 and 6. If the other implication is not true, then find the right definition(s) of a "somehow pointed" element.

Problem 10. Find the right "pointed" description for Hobby-McKenzie terms (omitting types **1** and **5**).

The following theorems about simple Taylor algebras and their subdirect products may be useful:

Theorem 11. *Let $\mathbf{A}$ be a finite, simple, idempotent Taylor algebra. Then at least one of the following holds:*

(1) *$\mathbf{A}$ has a proper absorbing subuniverse,*
(2) *The set of all pointed elements $Pt(\mathbf{A})$ is all of A,*
(3) *$\mathbf{A}$ is Abelian.*

Theorem 12. *Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be finite, simple, idempotent, algebras each of which is not absorbed by any proper subuniverse, and let $\mathbf{R} \leq_{sd} \mathbf{A}_1 \times \ldots \times \mathbf{A}_n$. Also assume that for all $1 \leq i < j \leq n$, $\pi_i \cap \pi_j = 1_R$ (here π_i is the kernel of the projection of R onto $\mathbf{A}_i$ and similarly π_j). In other words, we assume that R is "linked" (this is a related but not the same notion as the one used by Barto and Kozik, and also related to Zhuk's "connected"). Then either $R = \mathbf{A}_1 \times \ldots \times \mathbf{A}_n$, or there exists $I \subseteq \{1, 2, \ldots, n\}$ such that $|I| \geq 3$, $R = R_I \times \prod_{i \notin I} \mathbf{A}_i$ (here R_I denotes the projection of R onto $\prod_{i \in I} \mathbf{A}_i$) and such that $\mathbf{R}_I$ is Abelian.*

<u>Meta-questions</u>: In the case of the Hobby-McKenzie terms, congruence join-semidistributivity and congruence n-permutability, find a similar set of conditions. One could consider the following relaxations:

(1) Relax $\pi_i \cap \pi_j = 1_R$;
(2) Relax "simple";
(3) Find a different kind of absorption.

References

[1] Barto, L., Kozik, M., Stanovský, D.: Maltsev conditions, lack of absorption, and solvability. Algebra Universalis **74** no. 1 (2015), 185–206.
[2] Wires, A., On finite Taylor algebras, submitted.