CSPs of Finite Commutative Idempotent Binars

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joint work with

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slides available at
https://github.com/williamdemeo/Talks

General Problem: Find Maltsev conditions that characterize the complexity of CSPs of universal algebras.

CSP Dichotomy Conjecture

For a (finite, idempotent) algebra $\mathbf{A}$...

$\mathrm{CSP}(\mathbf{A})$ is tractable $\iff$ $\mathbf{A}$ has a weak-nu term operation

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For a (finite, idempotent) algebra $\mathbf{A}$...

$\mathrm{CSP}(\mathbf{A})$ is tractable $\implies$ $\mathbf{A}$ has a weak-nu term operation ✓

The left-to-right direction is known.

General Problem: Find Maltsev conditions that characterize the complexity of CSPs of universal algebras.

CSP Dichotomy Conjecture

For a (finite, idempotent) algebra $\mathbf{A}$...

$\mathrm{CSP}(\mathbf{A})$ is tractable $\impliedby$ $\mathbf{A}$ has a weak-nu term operation (?)

The right-to-left direction is open.

General Problem: Find Maltsev conditions that characterize the complexity of CSPs of universal algebras.

CSP DICHOTOMY CONJECTURE

For a (finite, idempotent) algebra $\mathbf{A}$...

$$\mathrm{CSP}(\mathbf{A}) \text{ is tractable} \iff \mathbf{A} \text{ has a weak-nu term operation} \quad (?)$$

A weak near unanimity (weak-nu) term operation is one that satisfies

$$t(x,x,\dots,x) \approx x \quad \text{(idempotent)}$$

$$t(y,x,\dots,x) \approx t(x,y,\dots,x) \approx \cdots \approx t(x,x,\dots,y)$$

A *binary* operation $t(x,y)$ is weak-nu if

$$t(x,x) \approx x \quad \text{(idempotent)}$$

$$t(y,x) \approx t(x,y) \quad \text{(commutative)}$$

So let's try to prove (?) for commutative idempotent binars.

COMMUTATIVE IDEMPOTENT BINARS

A CIB is an algebra $\mathbf{A} = \langle A, \cdot \rangle$ satisfying $x \cdot y \approx y \cdot x$ and $x \cdot x \approx x$.

QUESTION

Is every finite commutative idempotent binar tractable?

First Example: a semilattice is an associative CIB.

Semilattices are tractable.

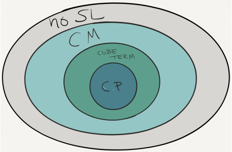
Pause to consider more general case for a minute...

GENERAL CASE

SOME WELL KNOWN FACTS

Let $\mathbf{A}$ be a finite idempotent algebra. Let $\mathbf{S}_2$ be the 2-elt semilattice.

$$\begin{aligned} \mathrm{V}(\mathbf{A}) \text{ is CP} &\iff \mathbf{A} \text{ has Malcev term} \\ &\implies \mathbf{A} \text{ has cube term} \\ &\implies \mathrm{V}(\mathbf{A}) \text{ is CM} \\ &\implies \mathbf{S}_2 \text{ is not in } \mathrm{V}(\mathbf{A}) \end{aligned}$$



RECENT RESULTS

$\mathbf{A}$ = a finite idempotent algebra
$\mathbf{S}_2$ = the 2-elt semilattice.

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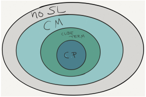

- cube term $\implies$ CM
  (Berman, Idziak, Marković, McKenzie, Valeriote, Willard 2010)
- CM $\implies$ $\mathbf{S}_2$ is not in $\mathrm{V}(\mathbf{A})$
  Proof: $\mathbf{S}_2 \in \mathrm{V}(\mathbf{A}) \Rightarrow \mathbf{S}_2^2 \in \mathrm{V}(\mathbf{A})$;
  $\mathrm{Con}\,(\mathbf{S}_2^2)$ is not modular.

Recent results

$\mathbf{A}$ = a finite **CIB**
$\mathbf{S}_2$ = the 2-elt semilattice.

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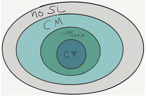
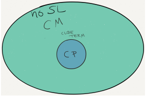


CIB case

Recent results

$\mathbf{A}$ = a finite **CIB**
$\mathbf{S}_2$ = the 2-elt semilattice.

$\mathrm{V}(\mathbf{A})$ is CP $\iff$ $\mathbf{A}$ has a Malcev term
$\implies$ $\mathbf{A}$ has a cube term
$\implies$ $\mathrm{V}(\mathbf{A})$ is CM
$\implies$ $\mathbf{S}_2$ is not in $\mathrm{V}(\mathbf{A})$
$\implies$ $\mathbf{A}$ has a cube term



CIB case

- 1st reduction by cube-term blockers.

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$\implies$ $\mathbf{A}$ has a cube term
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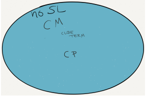


CIB case

- 1st reduction by cube-term blockers.
- 2nd reduction by Kearnes-Tschantz.

First Reduction
by Cube-Term Blockers

Marković, M. Maróti, McKenzie (M^4)
"Finitely related clones and algebras with cube terms" (2012)

A cube-term blocker (CTB) is a pair (C, B) of subuniverses satisfying $\emptyset < C < B \leqslant A$ and for every $t(x_1, \dots, x_n)$ there is an index $i \in [n]$ with

$$(\forall (b_1, \dots, b_n) \in B^n)(b_i \in C \longrightarrow t(b_1, \dots, b_n) \in C).$$

M^4 prove a finite idempotent algebra has a cube term iff it has no CTB.

Lemma

A finite CIB $\mathbf{A}$ has a CTB if and only if $\mathbf{S}_2 \in \mathrm{HS}(\mathbf{A})$.

Proof.

(C, B) a CTB implies $\theta = C^2 \cup (B - C)^2$ a congruence with $\mathbf{B}/\theta \cong \mathbf{S}_2$.

Conversely, suppose $\mathbf{S}_2 \in \mathrm{HS}(\mathbf{A})$, and $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ with $\mathbf{B}/\theta$ a meet-SL for some θ. Let C/θ be the bottom of $\mathbf{B}/\theta$, then (C, B) is a CTB. □

Second Reduction

Kearnes and Tschantz
"Automorphism groups of squares and of free algebras" (2007)

Lemma

If V is an idempotent variety that is not congruence permutable, then there are subuniverses U and W of $\mathbf{F} := \mathbf{F}_V\{x, y\}$ satisfying

1. $x \in U \cap W$
2. $y \in U^c \cap W^c$
3. $(U \times F) \cup (F \times W) \leqslant \mathbf{F}^2$

For CIB's, either U or W will be an ideal.

This implies a CTB and a semilattice.

Remaining Questions for Finite CIBs

Conclusion

Let $\mathbf{A}$ be a finite CIB. Then

$$\mathbf{S}_2 \notin \mathsf{HS}(\mathbf{A}) \text{ if and only if } \mathrm{V}(\mathbf{A}) \text{ is congruence permutable.}$$

(so $\mathrm{CSP}(\mathbf{A})$ tractable in this case)

Open Question

Let $\mathbf{A}$ be a finite CIB with $\mathbf{S}_2$ in $\mathsf{HS}(\mathbf{A})$. Is $\mathrm{CSP}(\mathbf{A})$ tractable?

Recall, if $\mathrm{V}(\mathbf{A})$ is $\mathrm{SD}_\wedge$, then $\mathrm{CSP}(\mathbf{A})$ is tractable.

Revised Question

Let $\mathbf{A}$ be a finite CIB with $\mathbf{S}_2$ in $\mathsf{HS}(\mathbf{A})$, and $\mathrm{V}(\mathbf{A})$ not $\mathrm{SD}_\wedge$.

Is $\mathrm{CSP}(\mathbf{A})$ tractable?

Examples

$\cdot$	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

t	0	1	2	3
0	0	0	0	0
1	0	1	3	2
2	0	3	2	1
3	0	2	1	3

Cliff's idea: replace basic binary operation with a term from $\mathrm{Clo}(\mathbf{A})$, say $t(x, y) = (x \cdot (x \cdot y)) \cdot (y \cdot (x \cdot y))$.

If $\langle A, t\rangle$ tractable, then so is $\mathbf{A} = \langle A, \cdot\rangle$.

$$\{t\} \subseteq \mathrm{Clo}(\mathbf{A}) \implies \mathrm{Rel}(\mathrm{Clo}(\mathbf{A})) \subseteq \mathrm{Rel}(\{t\})$$
$$\implies \mathrm{CSP}(\mathbf{A}) \leqslant_P \mathrm{CSP}\langle A, t\rangle$$

$\langle A, t\rangle$ tractable $\implies$ $\mathbf{A}$ tractable

Examples

$\cdot$	0	1	2	3
0	0	0	1	1
1	0	1	3	2
2	1	3	2	1
3	1	2	1	3

t_2	0	1	2	3
0	0	0	0	1
1	0	1	3	2
2	0	3	2	1
3	1	2	1	3

Let $t_2(x, y) = x \cdot (x \cdot (x \cdot (x \cdot y)) \cdot y \cdot (y \cdot (x \cdot y)).$

$\langle A, t_2\rangle$ tractable

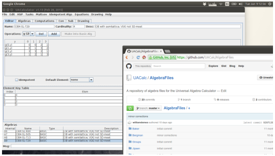
EXAMPLES

$\cdot$	0	1	2	3
0	0	0	2	1
1	0	1	3	2
2	2	3	2	1
3	1	2	1	3

Let $t_3(x,y) = \dots$?

Let $t_3(x,y,z) = \dots$?

...and about 25 others.



To see them, load UACalc with files from the Bergman directory at

`https://github.com/UACalc/AlgebraFiles`

Thank you for listening!