Determining congruence n-permutability is hard ($n \geq 3$?)

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Outline

Background

Hagemann & Mitschke 1973

Given $n \geq 2$, an algebra $\mathbf{A}$ generates a congruence n-permutable variety if and only if there exist ternary term operations $d_0, \ldots, d_n$ such that:

- $d_0(x, y, z) \approx x$,
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GEN-CLO$'$: Given a finite set A, a finite set of operations $\mathcal{F}$ on A, and a unary operation h on A, is $h \in \langle \mathcal{F} \rangle$?

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In this case say that g is CPB_0 (on coordinate i).

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5. For each $g \in \mathcal{U}$ (with arity n), define $t_g \colon B^{n+1} \to B$ to be

$$t_g(x_0, \ldots, x_n) = \begin{cases} g(x_1, \ldots, x_n) & \text{if } x_0 = h'(x_1) \\ 0 & \text{otherwise} \end{cases}$$

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9. Prove that generating a congruence n-permutable variety (for fixed $n \geq 3$) is satisfiable by CPB_0 operations.

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- If $\overline{x} \in A^m$ then $g'(\overline{x}) = p'(q_1'(\overline{x}), \ldots, q_n'(\overline{x})) = p(q_1(\overline{x}), \ldots, q_n(\overline{x})) = g(\overline{x})$.

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- If $\overline{x} \notin A^m$ then there is an i such that $q_i'(\overline{x}) = 0$, so $\overline{q'}(\overline{x}) \notin A^n$, therefore $g'(\overline{x}) = 0 = p'(\overline{q'}(\overline{x}))$.

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Therefore if $h \in \langle \mathcal{F} \rangle$ then $h' \in \langle \{f' \mid f \in \mathcal{F}\} \rangle \subseteq \langle \Gamma \rangle$.

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So If $h' \in \langle \Gamma \rangle$ then for every $g \in \mathcal{U}$, $g(x_1, \ldots, x_n) = t_g(h'(x_1), x_1, \ldots, x_n) \in \langle \Gamma \rangle$.

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- By idempotence, $v_1(x^m) = x$ for all $x \neq 0$ and so $v_0(x^m) = h'(x)$ for all $x \neq 0$.
- By Lemma then, $h \in \langle \mathcal{F} \rangle$.

Result

Definition

Say that an idempotent Mal'cev condition is **easily CPB-satisfiable** if there is a polynomial-time algorithm which takes as input a finite set A with distinguished element 0 and produces a set $\mathcal{U}$ of idempotent CPB_0 operations on A such that $\langle A, \mathcal{U} \rangle$ satisfies the Mal'cev condition.

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(In fact, this result is not restricted to Mal'cev conditions.)

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For fixed $n \geq 3$, determining whether or not finite algebra **A** generates a congruence n-permutable variety is EXPTIME-hard.

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For fixed $n \geq 3$, determining whether or not finite algebra $\mathbf{A}$ generates a congruence n-permutable variety is EXPTIME-hard.

Proof: Given a set A with distinguished element 0 and fixed $n \geq 3$ define idempotent ternary operations f_1, f_2 by

$$f_1(x,y,z) = \begin{cases} x & \text{if } y = z \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$f_2(x,y,z) = \begin{cases} z & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} .$$

Corollary

H 2013

For fixed $n \geq 3$, determining whether or not finite algebra $\mathbf{A}$ generates a congruence n-permutable variety is EXPTIME-hard.

Proof: Given a set A with distinguished element 0 and fixed $n \geq 3$ define idempotent ternary operations f_1, f_2 by

$$f_1(x, y, z) = \begin{cases} x & \text{if } y = z \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$f_2(x, y, z) = \begin{cases} z & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} .$$

Clearly this is a polynomial-time construction. Notice that $\langle A, \{f_1, f_2\}\rangle$ generates a congruence 3-permutable (and therefore congruence n-permutable) variety and that f_1 and f_2 are CPB_0. Therefore the preceding result applies and the Mal'cev condition of generating a congruence n-permutable variety is EXPTIME-hard. □

Consequences

Corollary

The following questions are EXPTIME-complete to answer with respect to a finite algebra **A**.

- Does **A** generate a $CD(n)$ variety (for fixed $n \geq 3$)?
- Does **A** generate a congruence distributive variety?
- Does **A** generate a congruence modular variety?
- Does **A** generate a congruence *n*-permutable variety (for fixed $n \geq 3$)?
- Does **A** generate a variety which omits types $\{1\}$? $\{1,2\}$? $\{1,5\}$? $\{1,2,5\}$? $\{1,4,5\}$? $\{1,2,4,5\}$?
- Does **A** support a weak near unanimity term operation of arity *n* (for fixed $n \geq 3$)?
- Does **A** support an idempotent cyclic term operation of arity *n* (for fixed $n \geq 3$)?
- Does **A** support a semilattice term operation?

Red text indicates H's 2013 additions to Freese & Valeriote's 2009 list.

Limitations

Definition

Let Γ be a set of columns of x's and y's of the same height, and let $\overline{v}$ be the column of the same height which consists entirely of x's. Say that $t\colon A^{\Gamma} \to A$ is a **Γ-special cube term** if

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A Mal'cev term is a term f satisfying the equations

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Note: For any particular Γ, TFAE

1. Possessing a Γ-special cube term is CPB-satisfiable,
2. Some projection is also a Γ-special cube term, and
3. Γ contains a column of x's.

Questions

Is it EXPTIME-complete to determine if an algebra has a Mal'cev term? A majority term? A near unanimity term? An edge term?

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Are there idempotent Mal'cev conditions which are CPB-satisfiable but not easily CPB-satisfiable?

Thank you!