The lattice of linear Mal'cev conditions

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Posets of Mal'cev conditions and interpretability types

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Interpretability form quasi-order. By a standard technique, we can get the corresponding partial order (we factor by equi-interpretability).

(Garcia, Taylor: The lattice of interpretability types of varieties, 1984.)

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Examples

- Mal'cev $\vee$ Jónsson terms $=$ Pixley term,
- Jónsson terms $\vee$ cube term $=$ near unanimity.
- Gumm terms $\vee$ SD($\vee$) $=$ Jónsson terms.

Meets

Meet of two abstract clones $\mathcal{A}$ and $\mathcal{B}$ is a clone $\mathcal{A} \times \mathcal{B}$ (the product in the category of clones) that is described by

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3. every algebra in $\mathcal{V}_1 \times \mathcal{V}_2$ is a product of an algebra from $\mathcal{V}_1'$ and an algebra from $\mathcal{V}_2'$.

Poset of linear Mal'cev conditions

A linear Mal'cev condition is a condition that do not include term composition, i.e., only equations of the form

$$f(x_{i_1}, \dots, x_{i_n}) \approx g(x_{j_1}, \dots, x_{i_m}), \quad \text{or} \quad f(x_{i_1}, \dots, x_{i_n}) \approx x_j$$

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But, the subposet is not a sublattice!

Meet of linear conditions

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Definition (Barto, Pinsker)

An algebra **A** is said to be a retract of **B** if there are two maps $a\colon B \to A$ and $b\colon A \to B$ such that $ab = 1_A$, and for every basic operation f we have

$$f_{\mathbf{A}}(a_1, \dots, a_n) = af_{\mathbf{B}}(b(a_1), \dots, b(a_n)).$$

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Observation

If $\mathbf{A}$ is a retract of $\mathbf{B}$ then $\mathbf{A}$ satisfies all the linear equations that $\mathbf{B}$ does.

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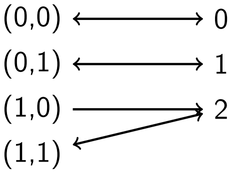
We choose algebra in $\mathcal{V}_1'$ that has no Jónsson terms, and similarly algebra in $\mathcal{V}_2'$ that has no Mal'cev term. For example

- $\mathbf{A} = (\{0,1\}, x+y+z, \mathrm{proj}_1^3, \mathrm{proj}_1^2)$, and
- $\mathbf{B} = (\{0,1\}, \mathrm{proj}_1^3, (x \vee y) \wedge (y \vee z) \wedge (x \vee z), \mathrm{proj}_2^2)$.

Meet of linear conditions (cont.)

Consider the interpretation of $\mathbf{A} \times \mathbf{B}$ in $\mathcal{W}$, and take its retract $\mathbf{C}$ via



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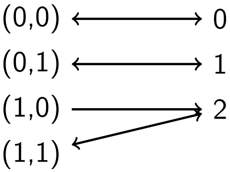

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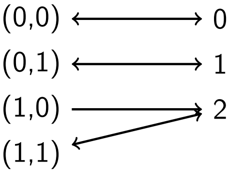

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1. Both $B' = \{0, 1\}$ and $A' = \{1, 2\}$ are subuniverses of $\mathbf{C}'$, $\operatorname{Clo} \mathbf{A}'$ is a reduct of $\operatorname{Clo} \mathbf{A}$, and $\operatorname{Clo} \mathbf{B}'$ is a reduct of $\operatorname{Clo} \mathbf{B}$,

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Problems with Mal'cev conditions

Lattice of linear varieties

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These problems can be solved by taking all linear varieties instead. (We lose Mal'cev conditions that are not strong.)

Prime elements of the lattice

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(Sequeira, Barto) Let X be a given set of variables, and $A \subseteq \mathrm{Eq}(X)$. We say that variety $\mathcal{V}$ is A-colorable if there is a map $c\colon F_{\mathcal{V}}(X) \to X$ such that

1. $c(x) = x$ for all $x \in X$, and
2. for every $\alpha \in A$ whenever $f \sim_{\hat{\alpha}} g$ then $c(f) \sim_{\alpha} c(g)$

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- congruence n-permutability,
- congruence modularity,
- satisfying non-trivial congruence identity,
- n-cube terms,
- triviality ($x \approx y$).

Prime elements of the lattice (cont.)

Theorem (Sequeira; Bentz-Sequeira)

Congruence modularity, n-permutability, satisfying non-trivial congruence identity, and n-cube term are prime with respect to varieties axiomatized by linear equations.

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A Mal'cev condition $\mathcal{M}$ satisfy coloring condition A if and only if for every linear variety $\mathcal{V}$ we have that either $\mathcal{V}$ satisfies $\mathcal{M}$, or $\mathcal{V}$ is interpretable in $\mathrm{Pol}(X, A)$

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for every basic operation f, and extend to terms.

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Theorem (Kiss, Kearnes, 2013)

Satisfying a non-trivial congruence identity and the set $\{\mathrm{Pol}(\mathbf{L}) : \mathbf{L} \textit{ is a semilattice}\}$ *split the lattice of idempotent varieties.*

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Thank you for your attention!