Cube-Absorption: Some Properties, Many Questions

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motivations

- Barto,Kozik - "Robust Satisfiability of Constraint Satisfaction Problems"
- Barto - "Dichotomy for Conservative Constraint Satisfaction Problems Revisited"
- A has cube term(edge term) iff no cube term blocker $C < D \leq A$

SD($\wedge$)	**Taylor**
near-unanimity	cube terms
absorption	
pointing terms	
pointed elements	

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- (C, D) cube term blocker: σ : terms $\to \mathbb{N}$,

$$f(d_1, ..., d_n) \in C \quad \text{when} \quad d_{\sigma(f)} \in C, \{d_1, ..., d_n\} \subseteq D$$

- $\langle\{0,1\},\wedge\rangle$

Better behaved Taylor algebras?

Cyclic terms make semilattices from cube term blockers (C, D):

- $\langle A, c\rangle$, $c(x_1, \ldots, x_n) \approx c(x_2, \ldots, x_n, x_1)$
- $\theta_C = (C \times C) \cup \Delta_D \in \operatorname{Con} D$, $d \in D \setminus C$
- $\operatorname{Clo}(C \cup \{d\})/\theta_C \approx \operatorname{Clo}\langle\{0, 1\}, \wedge\rangle$
- $\operatorname{cyc} A$ is generated by the cyclic terms of A

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Theorem

Let A be a finite idempotent Taylor algebra. Then

1. *$\operatorname{cyc} A$ has an edge term iff $\mathcal{V}(\operatorname{cyc} A)$ has Hobby-McKenzie terms ($\mathcal{V}(\operatorname{cyc} A)$ omits types $\{1,5\}$);*
2. *$\operatorname{cyc} A$ has an near-unanimity term iff $\mathcal{V}(\operatorname{cyc} A)$ is congruence join-semidistributive ($\mathcal{V}(\operatorname{cyc} A)$ omits types $\{1,2,5\}$).*

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Problem

- *Understand $\operatorname{cyc} A$ when type $\{5\} \subseteq \mathcal{V}(\operatorname{cyc} A) \subseteq \{3,4,5\}$.*
- *Understand $\operatorname{cyc} A$ when $\mathcal{V}(\operatorname{cyc} A)$ admits types $\{2,5\}$.*
- *When is $\operatorname{cyc} A$ finitely related; in particular, if A is finitely related, when is $\operatorname{cyc} A$ finitely related?*

Better behaved Taylor algebras?

$A =$

$\cdot$	0	1	a	b
0	0	1	b	1
1	1	0	1	1
a	b	1	a	a
b	1	1	a	b

- $\{0,1\}$ abelian
- $\{a,b\}$ semilattice
- A simple, type $A = 3$
- $\{0,1\} \lhd_{(xy)(zw)} A$

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$$A = \begin{array}{c|c|c|c|c} \cdot & 0 & 1 & a & b \\ \hline 0 & 0 & 1 & b & 1 \\ \hline 1 & 1 & 0 & 1 & 1 \\ \hline a & b & 1 & a & a \\ \hline b & 1 & 1 & a & b \end{array}$$

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Theorem

Let A be a finite idempotent Taylor algebra. Then either

- *A has proper absorption, or*
- *type* $(\alpha, 1_A) \subseteq \{2,3\}$ *for any maximal* $\alpha \in \operatorname{Con} A$.

Def I

Definition

Let $B \leq A$. We say B is *cube-absorbing* subset of A, and write $B \prec A$, if there exists a term $m(x_1,\dots,x_n)$ and tuples $u_1,\dots,u_k : \{x_1,\dots,x_n\} \to \{x,y\}$ such that for every $i \leq n$ there exists $j \leq k$ such that $u_j(x_i) = y$ and $m(u_i)[B,a] \subseteq B$ for each $a \in A$ and $i \leq k$. The maps u_i are *absorption identities* for $m(x_1,\dots,x_n)$.

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Example

Consider $B \prec_m A$ for $m(x_1, x_2, x_3, x_4, x_5)$ with absorption identities $u_1 = (xyyxy), u_2 = (xyxyx), u_3 = (yyyxx)$. Then $\forall a \in A$, the absorption inclusions $m(u_i)[B, a] \subseteq B$ are

$$\begin{aligned} m(B, a, a, B, a) &\subseteq B \\ m(B, a, B, a, B) &\subseteq B \\ m(a, a, a, B, B) &\subseteq B. \end{aligned}$$

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Example

- $B \triangleleft A$
- A has edge term iff every $a \prec A$ (finite idempotent)
- (C, D) cube-term blocker: $C \triangleleft_{\text{cyclic}} D$

Existence and Hereditary Characterization non-examples

Lemma

Let A be a finite idempotent algebra. If $\mathcal{V}(A)$ is a Taylor variety, then there exists a cube-absorbing pair $C \prec B \leq A$.

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Lemma

Let A be a finite idempotent algebra. If $\mathcal{V}(A)$ is a Taylor variety, then there exists a cube-absorbing pair $C \prec B \leq A$.

- Hereditary existence of cube-absorbing pairs ($\forall B \leq A$, $\exists C \prec D \leq B$) does not imply $\mathcal{V}(A)$ Taylor: On $\{1,2,3\}$ define $c(xyz), p(xyz), q(xyz)$ by
  - Set $2 = c(233) = c(332)$, $3 = c(322) = c(223)$ and let $c(xyz)$ be first projection otherwise
  - Set $3 = f(122) = f(133)$, $2 = g(122) = g(133)$ and let $f(xyz), g(xyz)$ be second projection otherwise.
- Hereditary existence of cube absorption ($\forall B \leq A$, $\exists C \prec B$) implies $\mathcal{V}(A)$ is Taylor
  - Converse: groupoid for directed 3-cycle

Hereditary Singleton Cube Absorption

- $\forall B \leq A, \exists a \in B, b \prec B$

Using the characterization of Mal'cev condition by Walter Taylor answering Grätzer:

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Theorem

1. *The class $\mathcal{K}$ of varieties in which every algebra has singleton cube-absorption is a Mal'cev condition.*
2. *The class $\mathcal{K}$ of varieties of algebras in which every singleton subalgebra is cube-absorbing is a Mal'cev condition.*
3. *The class $\mathcal{K}$ of varieties in which every finite algebra has singleton cube-absorption is a weak-Mal'cev condition.*

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3. *The class $\mathcal{K}$ of varieties in which every finite algebra has singleton cube-absorption is a weak-Mal'cev condition.*

- Is (3) actually a Mal'cev condition? Restricted to locally finite varieties?
- Find a "nice" explicit description of the Mal'cev conditions.

Hereditary Singleton Cube Absorption

$\forall B \leq A, \exists a \in B, \ b \prec B$

- $F_n = F_{\mathcal{V}(A)}(n)$ is too
- Find terms $r(x, y)$ and $e(\bar{x})$ such that

Hereditary Singleton Cube Absorption

$\forall B \leq A, \exists a \in B,\ b \prec B$

- $F_n = F_{\mathcal{V}(A)}(n)$ is too
- Find terms $r(x,y)$ and $e(\bar{x})$ such that
  - $r(x,y) \prec_{e(\bar{x})} F_2$ is edge-absorbing.
  - whenever $a \prec B \in \mathcal{V}$, $a = r(a,y)$, $\forall y \in B$
  - for all $f(x,y) \in F_2$,

    $$r(r(y,x), f(x,y)) \approx r(y,x) \quad \text{and} \quad r(x,y) \approx r(r(x,y), f(x,y))$$

  - $\{b : x \mapsto r(x,b) \text{ surjective }\} = A$ iff A has edge term

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Non-trivial subalgebras:

$\{a,b\}$	2-element semilattice with bottom a
$\{1,b\}$	2-element semilattice with bottom 1
$\{1,a\}$	2-element semilattice with bottom 1
$\{0,1\}$	2-element abelian group
$\{0,1,b\}$	with $\{0,1\}$ abelian absorbing

- $\{0,1\} \lhd_r A$ where $r(xyzw) = (xy)(zw)$
- $\{0\},\{1\} \prec_m A_4$ where $m = t * r$ with $t(x,y,z) := (xy)z$

Def II

We say B is *k-edge absorbing* if it is cube-absorbing with term $m(x_1, \ldots, x_k)$ and absorption identities

$$
\begin{aligned}
u_1 &= (y, y, x, x, x, \ldots, x) \\
u_2 &= (y, x, y, x, x, \ldots, x) \\
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&\vdots \\
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Theorem

Let A be a finite idempotent Taylor algebra. Then there exists a single term $m(\bar{x})$ such that $B \prec_{m(\bar{x})} D$ is edge-absorbing whenever $B \prec D \leq A$ and either B is a singleton or (B, D) is a cube-term blocker.

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Problem

Let A be a finite idempotent algebra. Then B is a cube-absorbing subalgebra of A iff B is an edge-absorbing subalgebra of A?

weakly compact d-quasi-representations

$B \prec A$ edge-absorbing

$$\begin{aligned} p(Baa) &\subseteq B & d(xy) &= p(xxy) \\ s(yxx...xx) &= p(xxy) & d(x, d(xy)) &= d(xy) \\ s(BAB...BB) &\subseteq B \\ &\vdots \\ s(BBB...BA) &\subseteq B \end{aligned}$$

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- def. from "Varieties with Few Subalgebras of Powers" works for any $d(x, d(xy)) = d(xy)$
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- $S \subseteq A^n$ weakly compact d-quasi-representation of R
- d associated to edge-term $\Rightarrow \mathrm{Sg}^{A^n}(S) = R$
- In general, what properties does $\mathrm{Sg}^{A^n}(S) < R$ have?
  - $d(x, y)$ "sees" all singleton $b \prec B \leq A$ and cube-term blockers (C, D)
  - walking subalgebras through subpowers: $R^+_{ij}[B]$, maximal components above singletons and blockers in appropriates quasi-orders (potato systems)

Thank You