Open Problems in Clone Theory

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Outline

Definition

Let A be a finite set.
O_A is the set of all operations on A.
A set of operations $F \subseteq O_A$ is a clone if F is closed under composition and contains all projections.
J_A is the set of all projections.

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Notations

- For a set of operations M, by $\mathrm{Clo}(M)$ we denote the minimal clone containing M.
- For a set of operations M, by $\mathrm{Inv}(M)$ we denote the set of all invariants of M.
- For a set of relations C, by $\mathrm{Pol}(C)$ we denote the set of all operations preserving every relation from C.

What is the main problem of Clone Theory?

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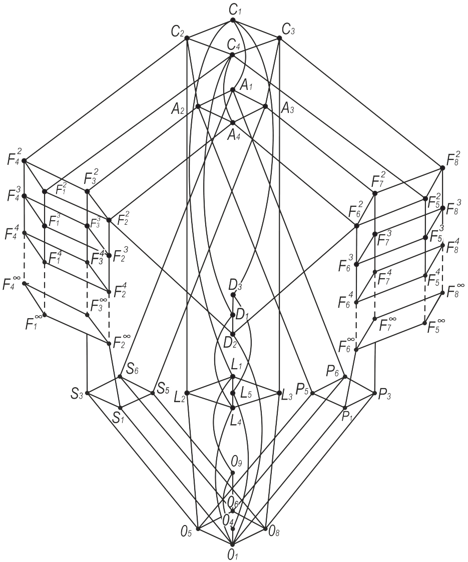
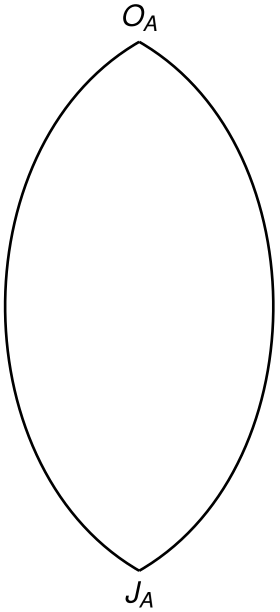
- To get a description of all clones!

The lattice of all clones on two elements(for $|A| = 2$)



Emil Post (1921, 1941)

For $|A| > 2$



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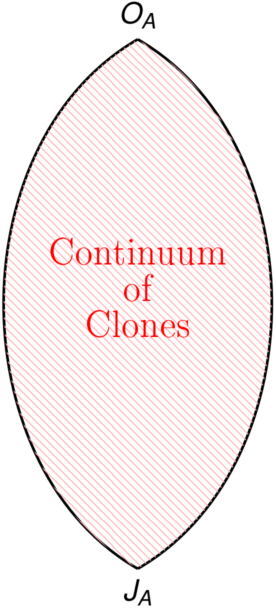

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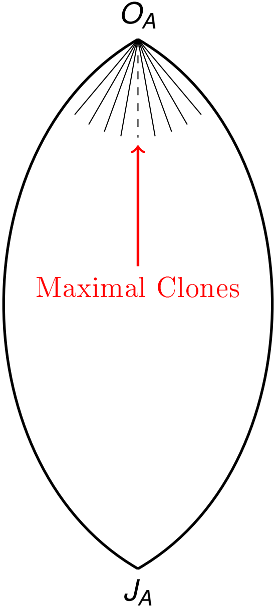

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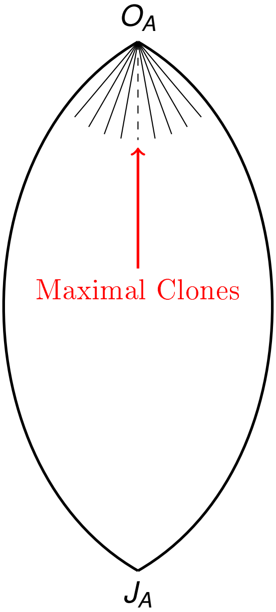

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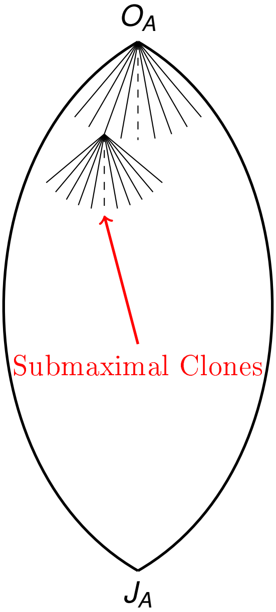

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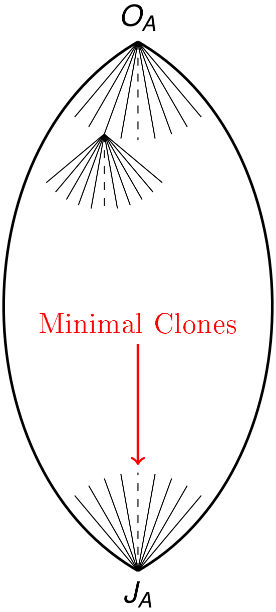

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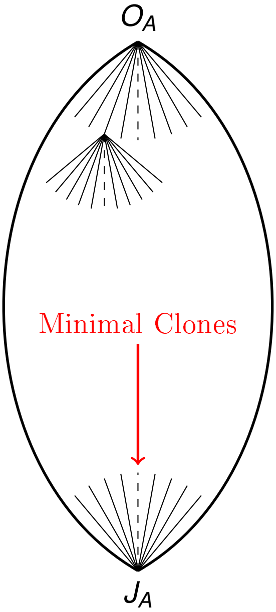

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- All minimal clones for $|A| = 3$ were found (B. Csákány, 1983)
- All minimal clones for $|A| = 4$ were found (Karsten Schölzer, 2012)

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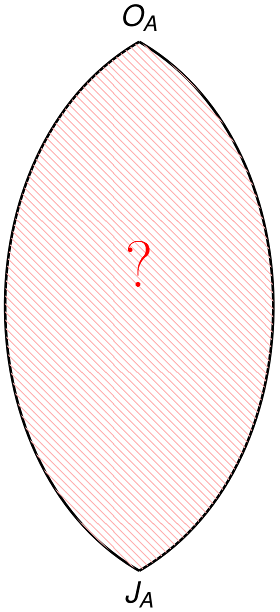

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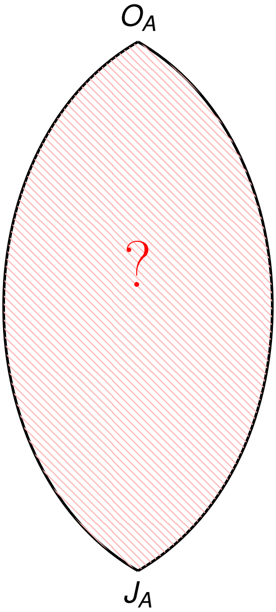
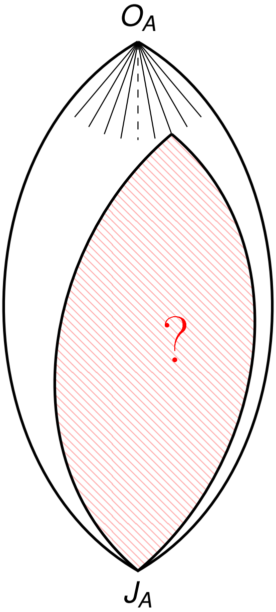


- Can we describe a significant part of the lattice?

For $|A| > 2$



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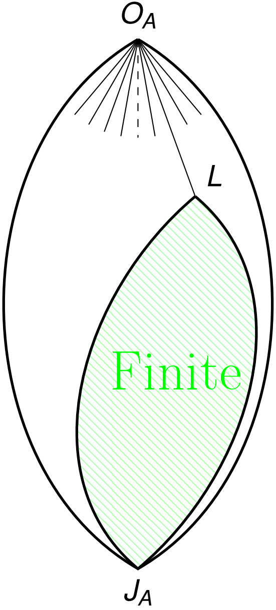

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- For the maximal clone of linear operations the lattice of subclones is finite and known ($|A|$ is a prime number) (A. A. Salomaa, 1964

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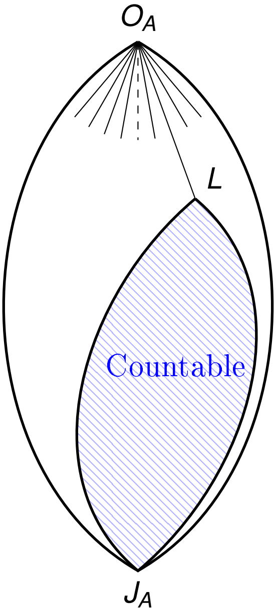

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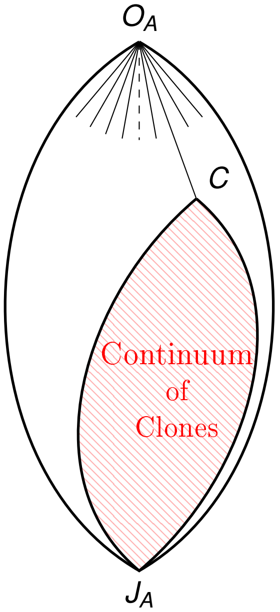

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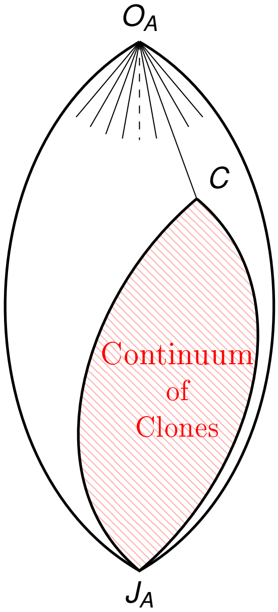

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Can we describe a sublattice that has continuum cardinality?

($|A| = 3$) The Maximal Clone of Self-Dual Operations

Here we assume that $A = \{0, 1, 2\}$.

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Definitions

An n-ary operation on three elements is called self-dual if

$$f(x_1 + 1, x_2 + 1, \ldots, x_n + 1) = f(x_1, x_2, \ldots, x_n) + 1$$

for all $x_1, x_2, \ldots, x_n \in \{0, 1, 2\}$.
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- Self-dual operations are operations that preserve the relation $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$.
- The set of all self-dual operations is a maximal clone on three elements

- Many clones of self-dual operations were found (S. S. Marchenkov, J. Demetrovich, L. Hannak, 1980).
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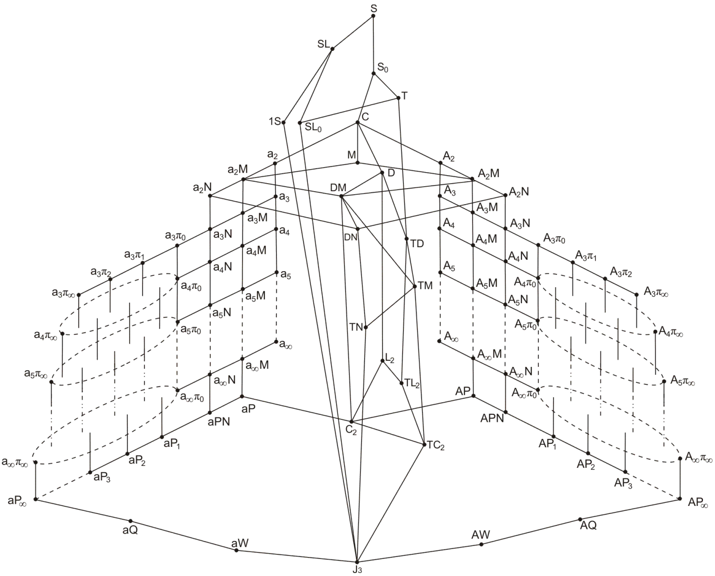
Main Result (D. Zhuk, 2010)

All clones of self-dual operations on three elements are described

- This is the first maximal clone besides the clone of all linear operations that has such description.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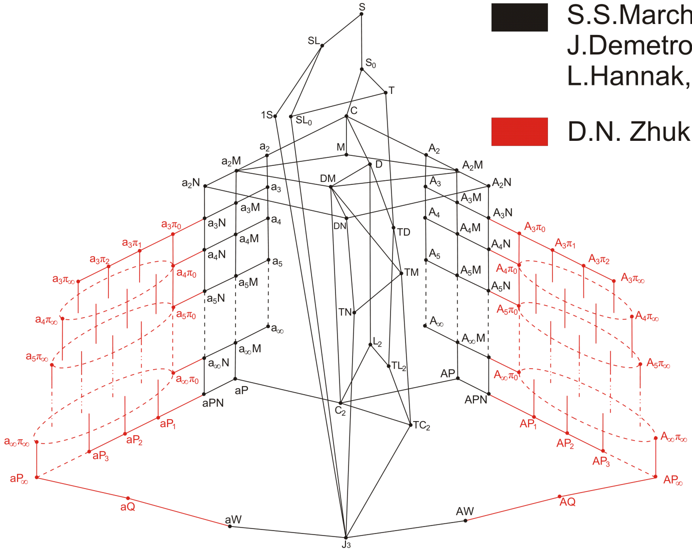
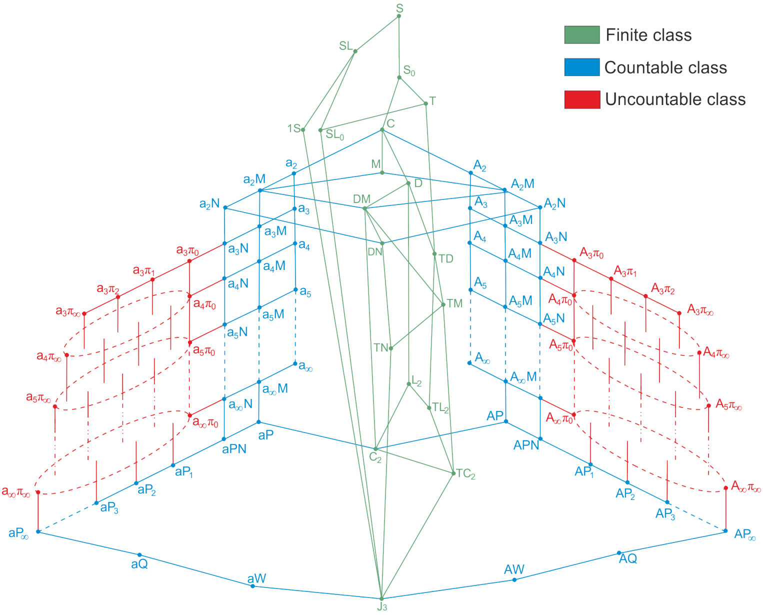


S.S.Marchenkov, J.Demetrovich, L.Hannak, 1980

D.N. Zhuk, 2010


Finite class
Countable class
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- If $F_1, F_2 \subseteq \Pi$ are nonempty down-sets, then $F_1 \neq F_2 \Rightarrow \mathrm{Pol}(F_1) \neq \mathrm{Pol}(F_2)$.
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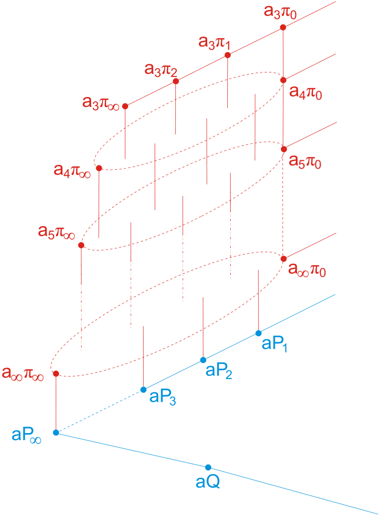
Why this description is complete and final?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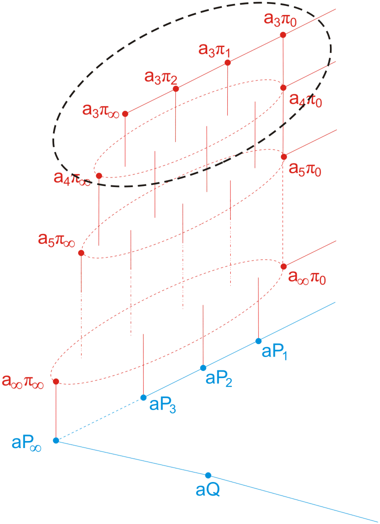
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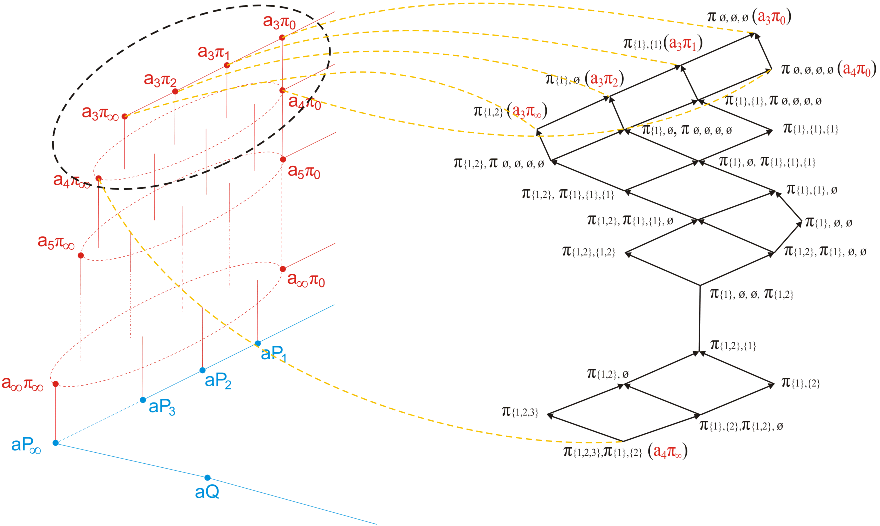
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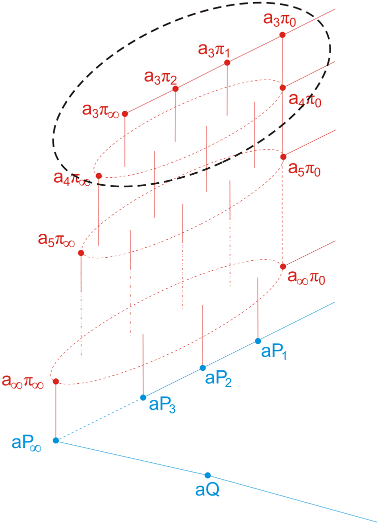

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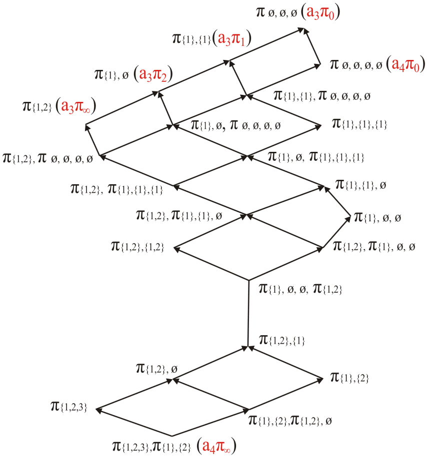

$\pi_{\emptyset,\emptyset,\emptyset}$ $(a_3\pi_0)$
$\pi_{\{1\},\{1\}}$ $(a_3\pi_1)$
$\pi_{\emptyset,\emptyset,\emptyset,\emptyset}$ $(a_4\pi_0)$
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$\pi_{\{1,2\}}$ $(a_3\pi_\infty)$
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$\pi_{\{1,2,3\}}$
$\pi_{\{1\},\{2\}}, \pi_{\{1,2\},\emptyset}$
$\pi_{\{1,2,3\}}, \pi_{\{1\},\{2\}}$ $(a_4\pi_\infty)$

Properties of the clones and the lattice

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A basis of a clone is a minimal by inclusion generating set.

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Relation degree of a clone is the minimal arity of relations such that the clone can be defined as the set of all operations that preserve these relations.

- The relation degree of each clone of self-dual operations is found
- The values of the relation degree prove that our description of clones is optimal. That is, these clones can not be defined by relations of smaller arity.

Definition

$\mathbb{L}_3$ is the set of all clones of self-dual operations,

$$\mathbb{L}_3^{\uparrow}(M) := \{M' \in \mathbb{L}_3 \mid M \subseteq M'),$$

$$\mathbb{L}_3^{\downarrow}(M) := \{M' \in \mathbb{L}_3 \mid M' \subseteq M).$$

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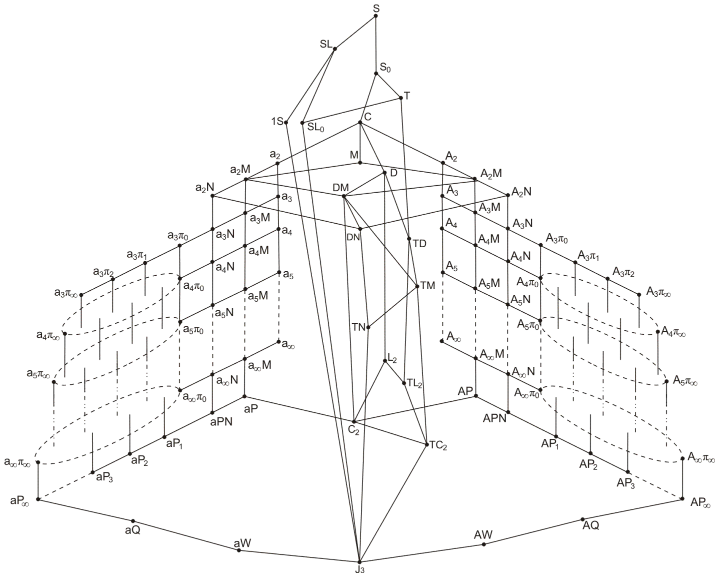
- For each clone $C \in \mathbb{L}_3$ the cardinality of $\mathbb{L}_3^{\uparrow}(C)$ is found.
- For each clone $C \in \mathbb{L}_3$ the cardinality of $\mathbb{L}_3^{\downarrow}(C)$ is found.
- $\mathbb{L}_3^{\downarrow}(\mathrm{Pol}(F))$ has continuum cardinality for every down-set $F \subseteq \Pi$ such that $F \neq \Pi$, $\mathbb{L}_3^{\downarrow}(\mathrm{Pol}(\Pi)) = 4$.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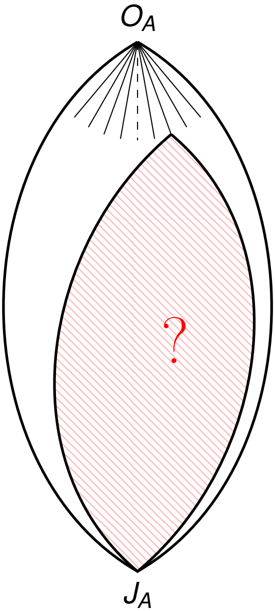
SL
S_0
T
1S
SL_0
C
a_2
M
A_2
a_2M
D
A_2M
a_2N
DM
A_2N
a_3
A_3
a_3M
A_3M
a_4
A_4
$a_3\pi_0$
a_3N
DN
TD
A_3N
$A_3\pi_0$
$a_3\pi_1$
a_4M
A_4M
$A_3\pi_1$
$a_3\pi_2$
a_4N
a_5
A_5
A_4N
$A_3\pi_2$
$a_3\pi_\infty$
$a_4\pi_0$
a_5M
TM
A_5M
$A_4\pi_0$
$A_3\pi_\infty$
a_5N
A_5N
$a_4\pi_\infty$
$a_5\pi_0$
TN
$A_5\pi_0$
$A_4\pi_\infty$
a_∞
A_∞
$a_5\pi_\infty$
$a_\infty M$
L_2
$A_\infty M$
$a_\infty N$
TL_2
$A_\infty N$
$A_5\pi_\infty$
$a_\infty\pi_0$
AP
$A_\infty\pi_0$
aP
aPN
APN
C_2
TC_2
aP_1
AP_1
$a_\infty\pi_\infty$
aP_2
AP_2
$A_\infty\pi_\infty$
aP_3
AP_3
aP_∞
AP_∞
aQ
AQ
aW
AW
J_3



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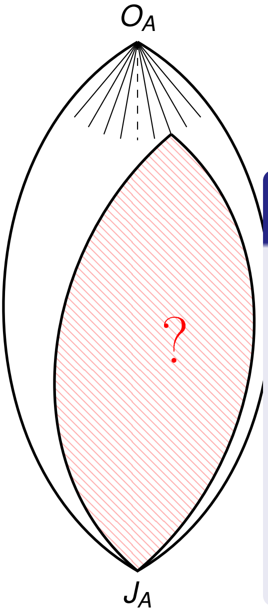
- Which maximal clone is the "smallest" one?



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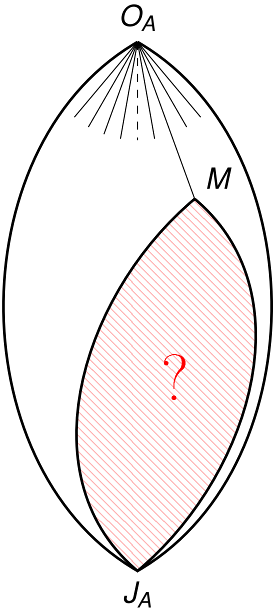
The number of clones containing a majority operation

Clone	Number of clones
$\mathrm{Pol}(0)$	1 744 466
$\mathrm{Pol}(\{0,1\})$	1 722 818
$\mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}$	36 942
$\mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 & 2 \end{pmatrix}$	1 546
$\mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$	17
$\mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 & 0 & 0 \end{pmatrix}$	84 201

Next steps



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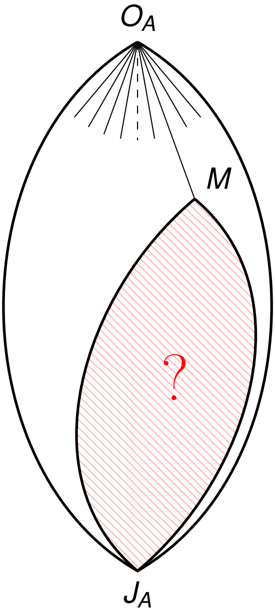

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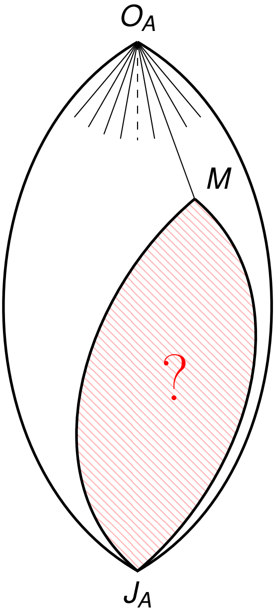

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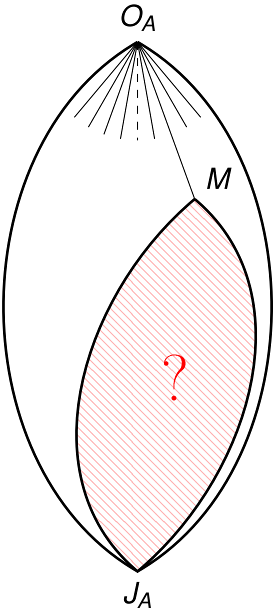

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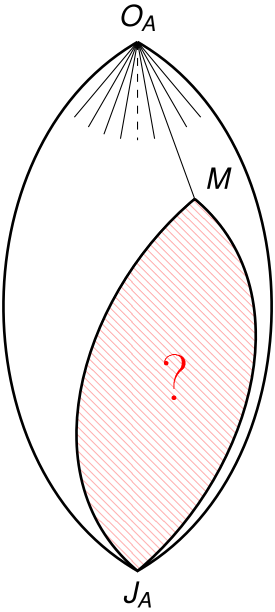

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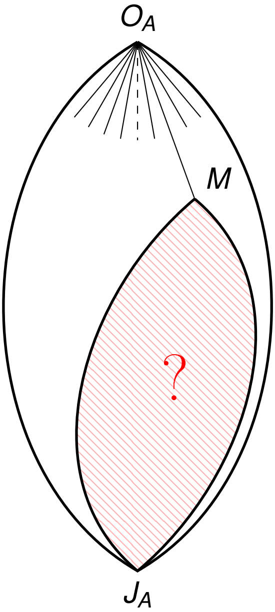

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The New Main Problems of Clone Theory

To find out how

- to check properties of clones
- to find finite parts of the lattice
- to check properties of the lattice of clones

How we define a clone

- Given a set of operations F. A clone is defined by $\mathrm{Clo}(F)$, that is the minimal clone containing F.

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- Given two sets of operations F_1 and F_2. A clone is defined by $\mathrm{Clo}(F_1) \cap \mathrm{Clo}(F_2)$.

General Problem 1

Decide whether two definitions give the same clone.

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Decision Problems

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4. Given 3 sets of operations B, C, and D. Decide whether $\mathrm{Clo}(B) \cap \mathrm{Clo}(C) = \mathrm{Clo}(D)$.

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Given a clone, decide how many clones contain this clone.

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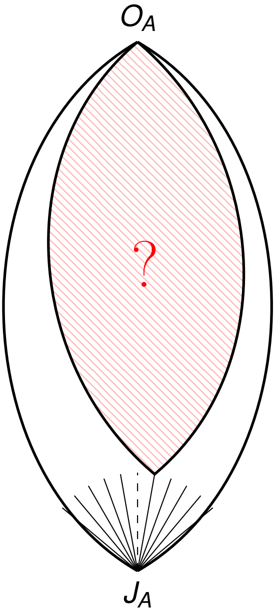
The principal filters for minimal clones

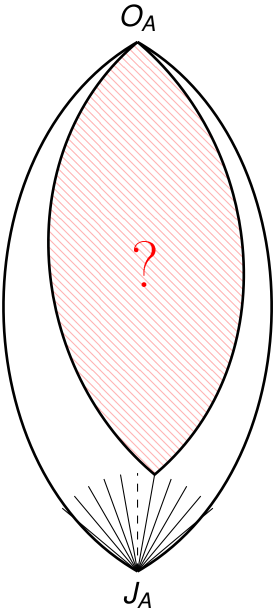


Definition

A clone $A \neq J_k$ is called minimal, if every operation from $A \setminus J_k$ generates this clone.

Examples of minimal clones

$\mathrm{Clo}(\{max\}), \mathrm{Clo}(\{x+1\})$.



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Question

How many clones contain a given minimal clone?

Theorem

Each minimal clone on k elements can be generated by f where

1. f is an unary operation.
2. f is a binary idempotent operation $f(a, a) = a$ for every $a \in E_3$.
3. f is a majority operation
   $\forall a, b \in E_3 : f(a, a, b) = f(a, b, a) = f(b, a, a) = a$
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   $\forall a, b \in E_3 : f(a, a, b) = f(a, b, a) = f(b, a, a) = b$
5. f is a semiprojection to the i-th variable
   $f \in P_k^n$, $3 \leq n \leq k$, for every tuple $(a_1, a_2, \ldots, a_n)$ satisfying $|\{a_1, a_2, \ldots, a_n\}| < n$ we have $f(a_1, a_2, \ldots, a_n) = a_i$.

We will assume that $i = 1$ for all occurring semiprojections.

Minimal Clones on three elements (found by B. Csákány)

There are exactly 84 minimal clones on 3 elements. One obtains every one of these clones by applying inner automorphisms of P_3 to exactly one of the following clones:

a) Minimal clones generated by unary operations

x	$\mathbf{c_0}$	$\mathbf{c_1}$	$\mathbf{c_2}$	$\mathbf{c_3}$
0	1	1	0	1
1	1	0	1	2
2	1	2	1	0

b) Minimal clones generated by idempotent binary operations

x	y	$\mathbf{b_1}$	$\mathbf{b_2}$	$\mathbf{b_3}$	$\mathbf{b_4}$	$\mathbf{b_5}$	$\mathbf{b_6}$	$\mathbf{b_7}$	$\mathbf{b_8}$	$\mathbf{b_9}$	$\mathbf{b_{10}}$	$\mathbf{b_{11}}$	$\mathbf{b_{12}}$
0	1	1	0	1	1	1	1	0	0	0	0	1	2
1	0	1	1	1	1	1	1	1	1	1	1	1	2
0	2	1	0	0	0	0	0	0	1	0	1	0	1
2	0	1	1	0	2	0	2	0	1	1	2	0	1
1	2	1	1	1	1	1	1	1	1	1	0	2	0
2	1	1	1	1	1	2	2	1	2	2	2	2	0

c) Minimal clones generated by majority operations $\mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3}$ and semiprojections $\mathbf{s_1}, \mathbf{s_2}, \mathbf{s_3}, \mathbf{s_4}, \mathbf{s_5}$

x	y	z	$\mathbf{m_1}$	$\mathbf{m_2}$	$\mathbf{m_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$
0	1	2	1	0	0	1	0	0	1	1
0	2	1	1	1	0	1	0	0	1	2
1	0	2	1	1	1	1	1	1	0	0
1	2	0	1	0	1	1	1	1	0	2
2	0	1	1	0	2	1	1	0	2	0
2	1	0	1	1	2	1	1	1	2	1

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- For the minimal clones generated by unary operations this problem was solved by J. Pantović and D. Voivodić (2000).
- The set of all clones containing a operation f has continuum cardinality for every $f \in \{\mathbf{s_1}, \mathbf{s_2}, \mathbf{b_1}, \mathbf{b_2}, \mathbf{b_4}, \mathbf{b_6}, \mathbf{b_8}, \mathbf{b_9}\}$. The set of all clones containing a operation g is at least countable for every $g \in \{\mathbf{s_3}, \mathbf{s_4}, \mathbf{b_3}, \mathbf{b_5}, \mathbf{b_7}, \mathbf{b_{10}}\}$ (J. Pantović, D. Voivodić, 2000).

Theorem (D. Zhuk, 2012)

Suppose a minimal clone on 3 elements M is generated by f. The set of all clones containing M

- is finite if $f \in \{\mathbf{c_3}, \mathbf{b_{12}}, \mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3}\}$,
- is countable if $f \in \{\mathbf{s_4}, \mathbf{s_5}\}$,
- has continuum cardinality if $f \in \{\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}, \mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, \dots, \mathbf{b_{10}}, \mathbf{b_{11}}, \mathbf{s_1}, \mathbf{s_2}, \mathbf{s_3}\}$.

Finite cases (except clones generated by majority operations)

$$\mathbf{c_3}(x) = x + 1;$$

$$\mathbf{b_{12}}(x, y) = 2x + 2y.$$

Countable cases

$$\mathbf{s_4}(x, y, z) = \begin{cases} 2x + 1, & \text{if } |\{x, y, z\}| = 3; \\ x, & \text{otherwise.} \end{cases};$$

$$\mathbf{s_5}(x, y, z) = \begin{cases} y, & \text{if } |\{x, y, z\}| = 3; \\ x, & \text{otherwise.} \end{cases}.$$

Minimal clones for $|A| > 3$

- All minimal clones on 4 elements were found (Pöshel, Kaluzhnin, Szczepara, Waldhauser, Jezek, Quackenbush, Karsten Schölzer, 2012).

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10. Find all minimal clones such that the corresponding principal filter is countable.

General problem 3

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- There exist finite sets of operations B and C such that $\mathrm{Clo}(B) \cap \mathrm{Clo}(C)$ is not finitely generated (L.Haddad, 1990).

Order of a clone C ($\mathrm{Ord}(C)$) is the minimal arity of a generating set.

A maximal clone of monotone operations is a clone $\mathrm{Pol}(\rho)$, where ρ is partial order relation with a greatest and a least element.

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1. Given partial order relation with a greatest and a least element. Decide whether $\mathrm{Pol}(\rho)$ is finitely generated. Open Problem 14: is this decidable?

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- For a clone C containing NU of arity $d+1$ we have $\mathrm{Ord}(C) \leq (|A|-1)^d - 1$, and this is a tight upper bound for sufficiently large $|A|$ (S. Kerkhoff, 2011).

Conjecture

Suppose ρ is a partial order relation with a greatest and a least element. Then $\mathrm{Pol}(\rho)$ is finitely generated iff it contains a NU.

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Suppose ρ is a partial order relation with a greatest and a least element. Then $\mathsf{Pol}(\rho)$ is finitely generated iff it contains a NU. Open Problem 15: Does this conjecture hold?

A clone C is called finitely related if $C = \mathrm{Pol}(\rho)$ for some relation ρ.

General problem 4

Given a clone. Decide whether the clone is finitely related. Find a relation that defines this clone.

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2. Given a finite set of operations M such that the clone $\mathrm{Clo}(M)$ is finitely related. Find a relation ρ such that $\mathrm{Pol}(\rho) = \mathrm{Clo}(M)$.

Whether a clone is finitely related

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Open Problems

18. Does there exist a minimal clone that is not finitely related?

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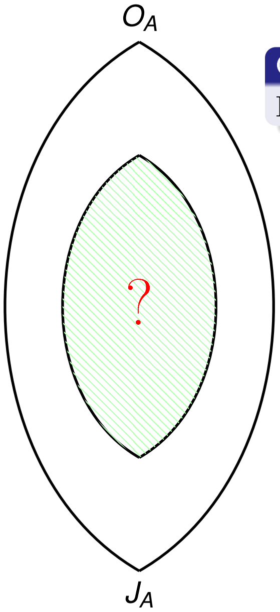
Open Problems

18. Does there exist a minimal clone that is not finitely related?
19. Does there exist a minimal clone on 3 elements that is not finitely related?

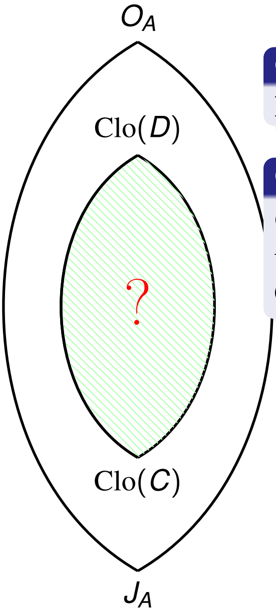
Finite Sublattices



General Problem 5

Find a sublattice between two clones if it is finite.

Finite Sublattices

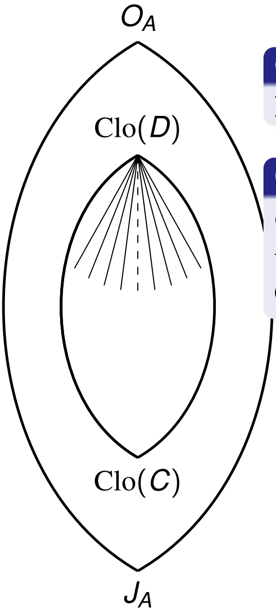


General Problem 5

Find a sublattice between two clones if it is finite.

Computational Problem

Given two finite sets of operations C and D such that $C \subseteq D$ and the sublattice between two clones Clo(C) and Clo(D) is finite. Find this sublattice.



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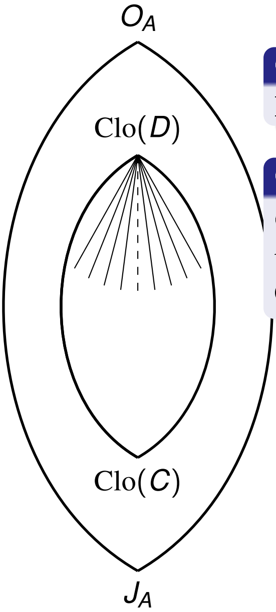
Given two finite sets of operations C and D such that $C \subseteq D$ and the sublattice between two clones Clo(C) and Clo(D) is finite. Find this sublattice.

- All clones in the sublattice are finitely generated.
- All maximal clones in Clo(D) can be found (each is defined as $\mathrm{Pol}(\rho)$ for some ρ).



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- All clones in the sublattice are finitely generated.
- All maximal clones in Clo(D) can be found (each is defined as Pol(ρ) for some ρ).
- What to do next? Open problem 21: is this problem decidable?

Computational Problem

Given two clones C and D such that $C \subseteq D$ and C contains a near-unanimity operation. Find the sublattice between C and D.

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Computational Problem

Given two clones C and D such that $C \subseteq D$ and C contains a near-unanimity operation. Find the sublattice between C and D.

Every clone containing NU of arity $m+1$ can be defined by relations of arity m.

Solution

- Consider all sets of relations of arity m preserving by a clone C.
- Find those of them that define different clones.

What is the complexity of this algorithm?

Near-Unanimity Case

Computational Problem

Given two clones C and D such that $C \subseteq D$ and C contains a near-unanimity operation. Find the sublattice between C and D.

Every clone containing NU of arity $m+1$ can be defined by relations of arity m.

Solution

- Consider all sets of relations of arity m preserving by a clone C.
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What is the complexity of this algorithm? Huge!

Computational Problem

Given two clones C and D such that $C \subseteq D$ and C contains a near-unanimity operation. Find the sublattice between C and D.

Every clone containing NU of arity $m+1$ can be defined by relations of arity m.

Solution

- Consider all sets of relations of arity m preserving by a clone C.
- Find those of them that define different clones.

What is the complexity of this algorithm? Huge!

Improvements(D.Zhuk, S.Moiseev, 2012)

- We consider only relations that cannot be represented as a conjunction of relations with smaller arities (these relations are called essential).
- We consider a closure operator on the set of essential relations.

Program that finds a sublattice

Input

- Two finite sets of relations M_{up} and M_{down} such that $M_{up} \subseteq M_{down}$.
- The arity N of near-unanimity operation preserving M_{down}.

Output

The lattice of clones between $\mathrm{Pol}(M_{down})$ and $\mathrm{Pol}(M_{up})$.

Program that finds a sublattice

Input

- Two finite sets of relations M_{up} and M_{down} such that $M_{up} \subseteq M_{down}$.
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The lattice of clones between $\mathrm{Pol}(M_{down})$ and $\mathrm{Pol}(M_{up})$.

- We consider all essential relations.
- Calculate the closure of every relation together with M_{up}.
- Choose minimal by inclusion sets: thus, we get maximal clones in $\mathrm{Pol}(M_{up})$.

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- Two finite sets of relations M_{up} and M_{down} such that $M_{up} \subseteq M_{down}$.
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Output

The lattice of clones between $\mathrm{Pol}(M_{down})$ and $\mathrm{Pol}(M_{up})$.

- We consider all essential relations.
- Calculate the closure of every relation together with M_{up}.
- Choose minimal by inclusion sets: thus, we get maximal clones in $\mathrm{Pol}(M_{up})$.
- Consider the closure of a maximal clone together with an essential relation.
- Choose the minimal by inclusion sets: thus, we get submaximal clones in $\mathrm{Pol}(M_{up})$.
- Repeat the procedure till we get M_{down}.

Example 1

Input

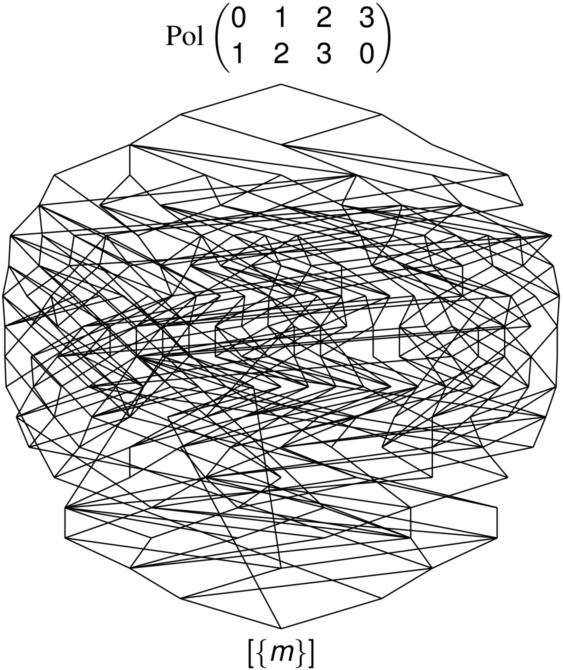
$A = \{0, 1, 2, 3\}$.

$M_{up} = \left\{ \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix} \right\}$

$M_{down} = \mathrm{Inv}(m)$, where $m(x, y, z) = \begin{cases} y, & \text{if } x = y; \\ x, & \text{otherwise.} \end{cases}$

$N = 3$.

Find the sublattice between $\mathrm{Clo}(m)$ and $\mathrm{Pol} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}$.



number of clones $= 199$

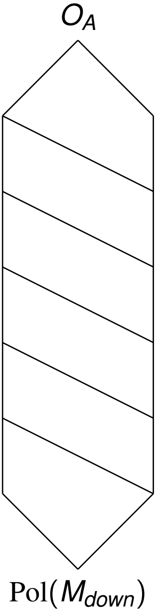
number of layers $= 19$

Example 2

Input

$A = \{0, 1\}$.
$M_{up} = \varnothing$
$M_{down} = \{\{0,1\}^n \setminus \{0\}^n \mid n \in \{1, 2, \ldots, 7\}\} \cup \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \right\}$.
$N = 8$.

Find the sublattice between $\mathrm{Pol}(M_{down})$ and $\mathrm{Pol}(M_{up})$.



$$\text{number of clones} = 14$$

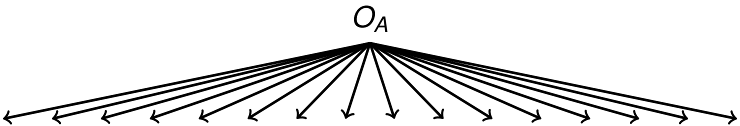
$$\text{number of layers} = 8$$

Input

$A = \{0, 1, 2\}$.
$M_{up} = \varnothing$
$M_{down} = \mathrm{Inv}(m)$, where m is a minimal majority operation.
$N = 3$.

Find all clones on 3 elements containing a majority operation.

Clones with a majority operation



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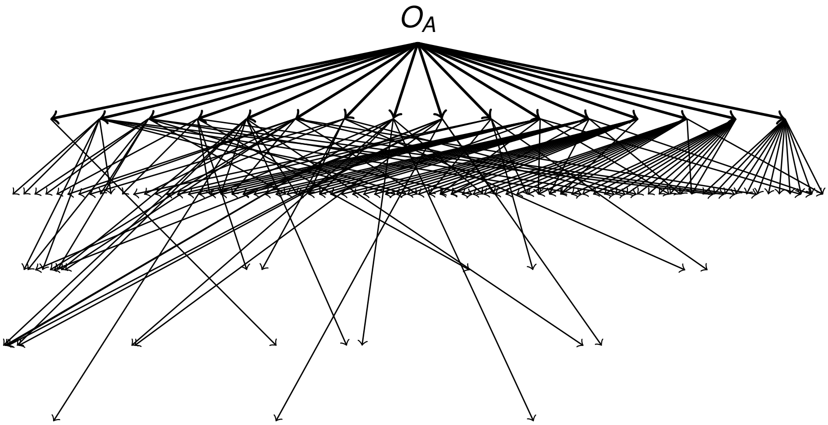

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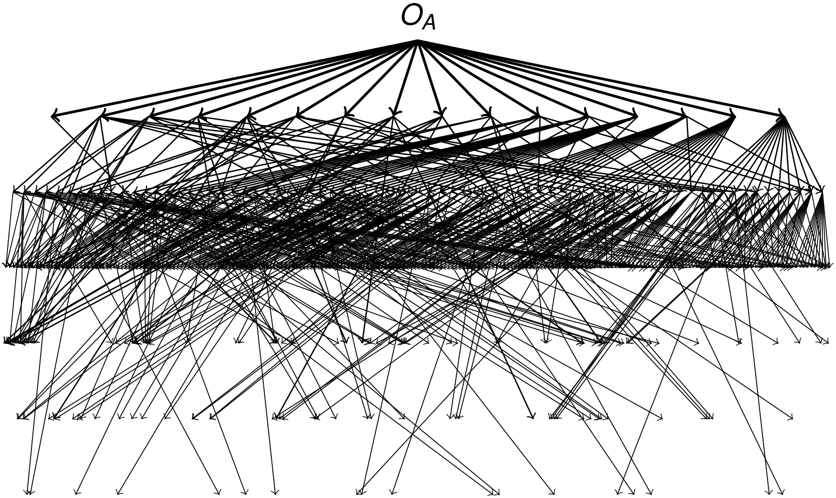

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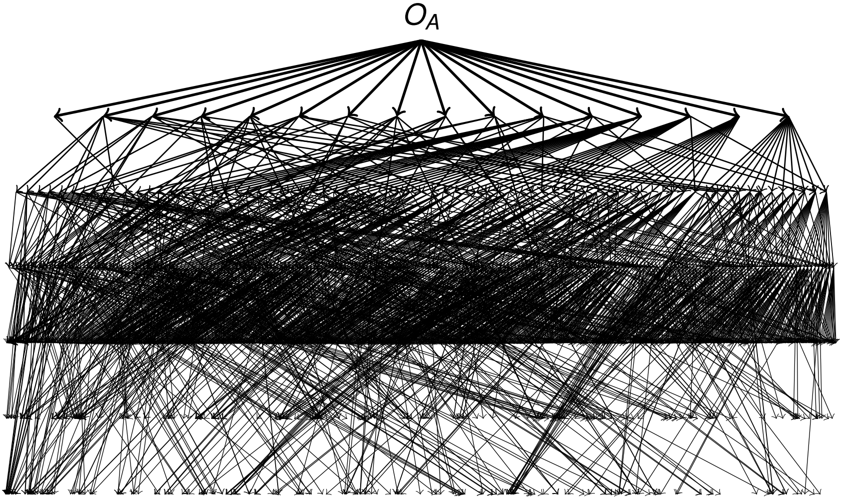
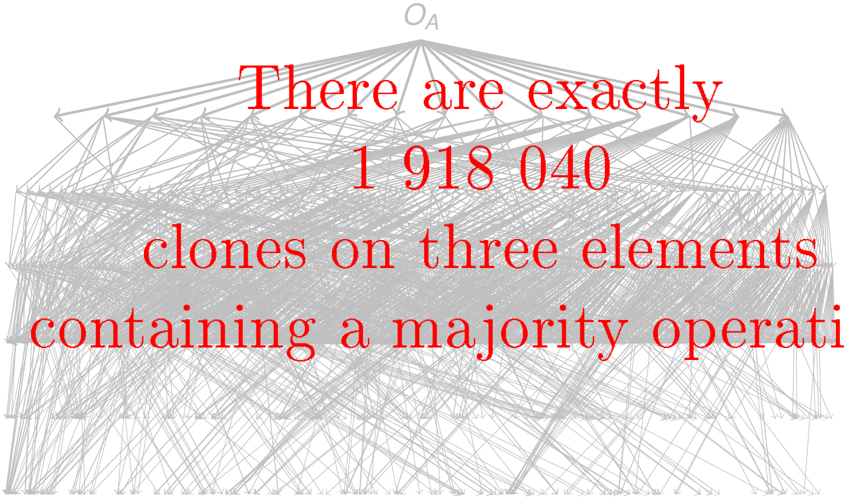
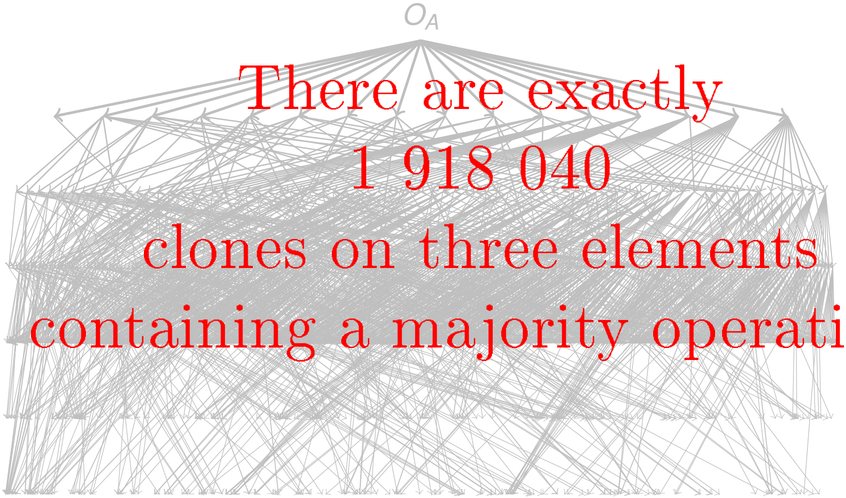




There are exactly
1 918 040
clones on three elements
containing a majority operation!



There are exactly
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clones on three elements
containing a majority operation!

We have three types of a minimal majority operation on 3 elements.

x_1	x_2	x_3	m_1	m_2	m_3
0	1	2	1	0	0
0	2	1	1	1	0
1	0	2	1	1	1
1	2	0	1	0	1
2	0	1	1	0	2
2	1	0	1	1	2

The Number of Clones and the number of Layers

	m_1	m_2	m_3
number of clones	928 767	325 581	99 152
number of layers	51	45	33

The size of a maximal chain

General Problem 6

What can be the size of a maximal chain in the lattice of clones?

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Partial Results

- If $|A| = 2$ then a maximal chain can be of size $5, 6, 7, \aleph_0$

The size of a maximal chain

General Problem 6

What can be the size of a maximal chain in the lattice of clones?

Partial Results

- If $|A| = 2$ then a maximal chain can be of size $5, 6, 7, \aleph_0$
- For prime $|A|$ the shortest maximal chain is of size 5. For composite $|A|$ the shortest maximal chain is of size ≥ 6 (Ágnes Szendrei).

The size of a maximal chain

General Problem 6

What can be the size of a maximal chain in the lattice of clones?

Partial Results

- If $|A| = 2$ then a maximal chain can be of size $5, 6, 7, \aleph_0$
- For prime $|A|$ the shortest maximal chain is of size 5. For composite $|A|$ the shortest maximal chain is of size ≥ 6 (Ágnes Szendrei).
- If $|A| = 3$ then there exists a maximal chain of size 52. This chain contains a minimal majority operation.

General Problem 6

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Open Problems

22. Does there exist a finite maximal chain of size greater than 52 on 3 elements?

General Problem 6

What can be the size of a maximal chain in the lattice of clones?

Partial Results

- If $|A| = 2$ then a maximal chain can be of size $5, 6, 7, \aleph_0$
- For prime $|A|$ the shortest maximal chain is of size 5. For composite $|A|$ the shortest maximal chain is of size ≥ 6 (Ágnes Szendrei).
- If $|A| = 3$ then there exists a maximal chain of size 52. This chain contains a minimal majority operation.

Open Problems

22. Does there exist a finite maximal chain of size greater than 52 on 3 elements?
23. Is the size of a finite maximal chain restricted?

General Problem 6

What can be the size of a maximal chain in the lattice of clones?

Partial Results

- If $|A| = 2$ then a maximal chain can be of size $5, 6, 7, \aleph_0$
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- If $|A| = 3$ then there exists a maximal chain of size 52. This chain contains a minimal majority operation.

Open Problems

22. Does there exist a finite maximal chain of size greater than 52 on 3 elements?
23. Is the size of a finite maximal chain restricted?
24. Is it true that a finite maximal chain of the maximal size always contains a minimal majority operation?

Open Problems

25. Find a set of relations C of arity at most m such that $\mathrm{Ord}(\mathrm{Pol}(C)) > |A|^m$ (or $> 2^{2^m}$).

Open Problems

25. Find a set of relations C of arity at most m such that $\mathrm{Ord}(\mathrm{Pol}(C)) > |A|^m$ (or $> 2^{2^m}$).
26. Find a set of operations M of arity at most m such that the relational degree of $\mathrm{Clo}(M)$ is greater than $|A|^m$ (or $> 2^{2^m}$).

Are there any undecidable problems in Clone Theory?

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Theorem (R.McKenzie, 1997)

It is undecidable for a finite algebra $\mathbb{A}$ and two elements $a, b \in A$ whether $\mathbb{A}$ has a term operation that is a near-unanimity operation on $\{a, b\}$.

Theorem (M.Maróti, 2000)

It is undecidable for a finite algebra $\mathbb{A}$ and two elements $a, b \in A$ whether $\mathbb{A}$ has a term operation that is a near-unanimity operation on $A \setminus \{a, b\}$.

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Let me know if you find other examples!

Thank you for your attention