The Variety Generated by $\mathbb{A}(\mathcal{T})$ – Two Counterexamples

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Park's Conjecture

Conjecture (Park)

If $\mathcal{V}$ is a finitely generated variety with finite residual bound, then $\mathcal{V}$ is finitely based.

Known if $\mathcal{V}$...

- is congruence distributive (Baker).
- is congruence modular (McKenzie).
- is congruence meet-semidistributive (CSD($\wedge$)) (Willard).
- has a difference term (Kearnes, Szendrei, Willard).

Finite Basis Theorems

Theorem (Baker)

If $\mathcal{V}$ is a finitely generated congruence distributive variety, then $\mathcal{V}$ is finitely based.

Some alternate proofs use:

- definable principal subcongruences
- bounded Maltsev depth

Theorem (Willard)

If $\mathcal{V}$ is a finitely generated congruence meet-semidistributive and has finite residual bound, then $\mathcal{V}$ is finitely based.

Do alternate proofs exist that use:

- definable principal subcongruences?
- bounded Maltsev depth?

Question

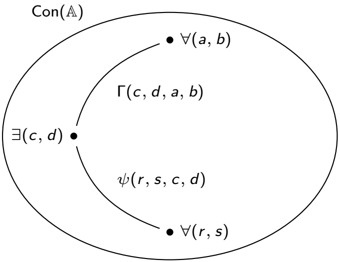
If $\mathbb{A}$ generates a variety that has finite residual bound and is CSD($\wedge$), does the variety have...

- *definable principal subcongruences?*
- *bounded Maltsev depth?*

Definable Principal Subcongruences

Definition

A variety $\mathcal{V}$ is said to have *definable principal subcongruences (DPSC)* if there are congruence formulas $\Gamma(\text{-},\text{-},\text{-},\text{-})$ and $\psi(\text{-},\text{-},\text{-},\text{-})$ such that for all $\mathbb{A} \in \mathcal{V}$ and all $a, b \in A$ there exist $c, d \in A$ such that $\Gamma(c, d, a, b)$ witnesses $(c, d) \in \mathrm{Cg}^{\mathbb{A}}(a, b)$ and $\psi(-, -, c, d)$ defines $\mathrm{Cg}^{\mathbb{A}}(c, d)$.



If $\mathcal{V}$ has DPSC and $\mathrm{Cg}^{\mathbb{A}}(a, b)$ is an atomic congruence of some $\mathbb{A} \in \mathcal{V}$, then $\psi(-, -, a, b)$ defines $\mathrm{Cg}^{\mathbb{A}}(a, b)$.

Definable Principal Subcongruences

If $\mathcal{V}$ has DPSC, then every atomic congruence is defined by ψ.

This means...

- there is $N \in \mathbb{N}$ such that
  - for all $\mathbb{A} \in \mathcal{V}$,
  - all $a, b \in A$ with $\mathrm{Cg}^{\mathbb{A}}(a, b)$ atomic, and
  - all $(c, d) \in \mathrm{Cg}^{\mathbb{A}}(a, b)$

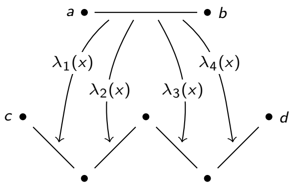
  there is $\mathbb{B} \leq \mathbb{A}$ of size at most N with $(c, d) \in \mathrm{Cg}^{\mathbb{B}}(a, b)$.

- to disprove DPSC for $\mathcal{V}$, find $\mathbb{A} \in \mathcal{V}$ and atomic $\mathrm{Cg}^{\mathbb{A}}(a, b)$ such that there is no bound on minimal $\mathbb{B} \leq \mathbb{A}$ witnessing $(c, d) \in \mathrm{Cg}^{\mathbb{B}}(a, b)$.

Bounded Maltsev Depth

Definition

A variety $\mathcal{V}$ has *Maltsev depth N* if for every $\mathbb{A} \in \mathcal{V}$ and every $a, b \in \mathbb{A}$, $(c, d) \in \mathrm{Cg}^{\mathbb{A}}(a, b)$ is witnessed by a Maltsev chain with associated polynomials of compositional depth at most N (and N is minimal).



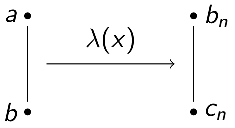
$\forall \mathbb{A} \in \mathcal{V}$, $\forall a, b, c, d \in A$...

$\lambda_i(x)$ has uniformly bounded depth.

The Strategy

- Find $\mathcal{V}$ that is CSD($\wedge$) and has finite residual bound.

- Find family of $\mathbb{B}_n \in \mathcal{V}$ and $a, b, b_n, c_n \in B_n$ such that...



  - $\mathrm{Cg}^{\mathbb{B}_n}(a, b)$ is atomic.
  - This is the "best" Maltsev chain witnessing $(b_n, c_n) \in Cg^{\mathbb{B}}(a, b)$.
  - The number of parameters used in $\lambda(x)$ scales with n.
  - $\lambda(x)$ has depth that scales with n.

- Letting n grow will demonstrate that $\mathcal{V}$ has neither DPSC nor bounded Maltsev depth.

The Algebra $\mathbb{A}(\mathcal{T})$

- McKenzie associated to each Turing machine $\mathcal{T}$ an algebra $\mathbb{A}(\mathcal{T})$ such that $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ has finite residual bound iff $\mathcal{T}$ halts.

- $\mathbb{A}(\mathcal{T})$ has a semilattice operation, so $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ is CSD($\wedge$). When $\mathcal{T}$ halts, $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ has finite residual bound.

- Willard proved that $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ is finitely axiomatizable iff $\mathcal{T}$ halts.

- By adding another operation to $\mathbb{A}(\mathcal{T})$, I proved that $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has DPSC iff $\mathcal{T}$ halts.

- Is $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ a counterexample to the claim that all such algebras have DPSC? (Yes)

$\mathbb{A}(\mathcal{T})$

For a Turing machine $\mathcal{T}$ with n states, the underlying set of $\mathbb{A}'(\mathcal{T})$ has $(20n + 8)$ elements:

$$A'(\mathcal{T}) = \{0, 1, 2, H, C, D, \partial C, \partial D, \\ C^s_{ir}, D^s_{ir}, M^r_i, \partial C^s_{ir}, \partial D^s_{ir}, \partial M^r_i \mid 0 \leq i \leq n \text{ and } r, s \in \{0, 1\}\}.$$

$\mathbb{A}'(\mathcal{T})$ has operations to emulate computation on certain tuples of the indexed elements:

$$\mathcal{L} = \{L_{irt} \mid \mathcal{T} \text{ has instruction } (\mu_i, r, s, L, \mu_j) \text{ and } t \in \{0, 1\}\},$$
$$\mathcal{R} = \{R_{irt} \mid \mathcal{T} \text{ has instruction } (\mu_i, r, s, R, \mu_j) \text{ and } t \in \{0, 1\}\}.$$

The operations of $\mathbb{A}(\mathcal{T})$ are

$$\left\{0, \wedge, (\cdot), J, J', S_0, S_1, S_2, T, I, F, U^0_F, U^1_F \mid F \in \mathcal{L} \cup \mathcal{R}\right\}.$$

The algebra $\mathbb{A}'(\mathcal{T})$ (has DPSC iff $\mathcal{T}$ halts) includes an operation K.

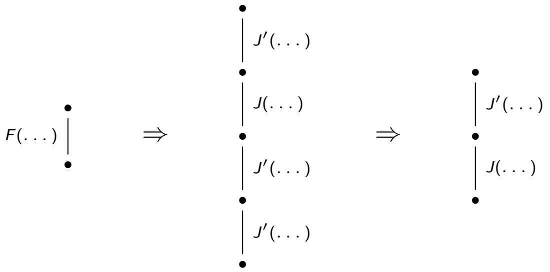
Let $\mathbb{A}'(\mathcal{T})$ be $\mathbb{A}(\mathcal{T})$ with a new operation, K, added to the language.

Theorem

The following are equivalent:

- $\mathcal{T}$ *halts.*
- $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ *has finitely many SI's, all finite.*
- $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ *has DPSC.*
- $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ *is finitely axiomatizable.*

Proof involves analysis of Maltsev chains:



The Counterexample

Define $\mathbb{B}_n = \langle\{a, b_i, d_i \mid 2 \le i \le n\}\rangle$ where

$$b_i = (D, D, \ldots, \overset{i}{\hat{D}}, 0, \ldots, 0), \qquad a = b_1,$$

$$d_i = (D, \ldots, D, \overset{i}{\partial\hat{D}}, 0, \ldots, 0), \quad c_i = (0, D, \ldots, \overset{i}{\hat{D}}, 0, \ldots, 0).$$

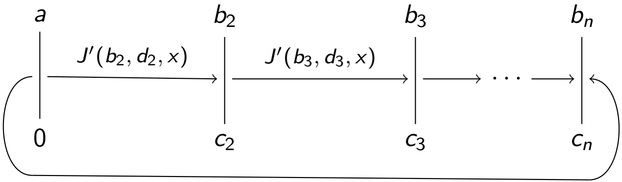
$\mathbb{A}(\mathcal{T})$ has **lots** of operations, but the only nonzero ones on $\mathbb{B}_n$ are

$$x \wedge y = \begin{cases} x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \qquad S_2(u, v, x, y, z) = \begin{cases} (x \wedge y) \vee (x \wedge z) & \text{if } u = \partial v \\ 0 & \text{otherwise} \end{cases}$$

$$J(x, y, z) = \begin{cases} x & \text{if } x = y \\ x \wedge z & \text{if } x = \partial y \\ 0 & \text{otherwise} \end{cases} \qquad J'(x, y, z) = \begin{cases} x \wedge z & \text{if } x = y \\ x & \text{if } x = \partial y \\ 0 & \text{otherwise} \end{cases}$$

Look at $(b_n, d_n) \in \mathrm{Cg}^{\mathbb{B}_n}(a, 0)$.

The Calculation



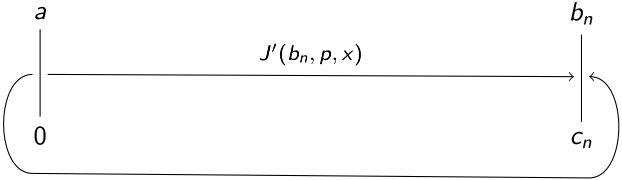
$$\lambda(x) = J'(b_n, d_n, J'(b_{n-1}, d_{n-1}, \cdots J'(b_2, d_2, x) \cdots))$$

- $\lambda(x)$ uses $2(n-1)$ distinct parameters (so the smallest $\mathbb{B} \leq \mathbb{B}_n$ witnessing $(b_n, c_n) \in \mathrm{Cg}^{\mathbb{B}_n}(0, a)$ is of size $\geq 2(n-1)$).
  Therefore $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ doesn't have DPSC.

- The compositional depth of $\lambda(x)$ is $n-1$.
  Therefore $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ doesn't have bounded Maltsev depth.

What about the K operation?

With K in the language, $\mathbb{B}_n$ contains an element p such that...



$$\lambda(x) = J'(b_n, d_n, J'(b_{n-1}, d_{n-1}, \cdots J'(b_2, d_2, x) \cdots))$$

The K operation was introduced precisely so that things like $\lambda(x)$ could be simplified.

Thank you.