Naturally dualizable algebras omitting types **1** and **5** have a cube term

(the talk formerly known as "Idempotent congruence modular algebras that admit a natural duality have cube terms")

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What?

Theorem

Let $\mathbb{A}$ be a finite algebra such that $\mathcal{V}(\mathbb{A})$ omits types $\mathbf{1}$ and $\mathbf{5}$. If $\mathbb{A}$ admits a natural duality, then $\mathbb{A}$ has a cube term.

- Naturally dualizable
- Omitting types $\mathbf{1}$ and $\mathbf{5}$
- Cube term

Natural dualities

Motivational examples to think of:

- Stone duality between Boolean algebras and Stone spaces
- Priestley duality between distributive lattices and bounded Priestley spaces
- Pontryagin duality for abelian groups

Ingredients for a duality:

- a finite algebra $\mathbb{A} = \langle A; F \rangle$
- a structured topological space $\underset{\sim}{\mathbf{A}} = \langle A; G, H, R, \mathcal{T} \rangle$ (G = total ops, H = partial ops, R = relations, $\mathcal{T}$ = discrete topology)
- If $\mathbb{B} \in \mathbf{SP}(\mathbb{A})$ then $\mathbb{B}^\partial = \mathrm{Hom}(\mathbb{B}, \mathbb{A}) \in \mathbf{S}_c\mathbf{P}_+(\underset{\sim}{\mathbf{A}})$
- If $\underset{\sim}{\mathbf{B}} \in \mathbf{S}_c\mathbf{P}_+(\underset{\sim}{\mathbf{A}})$ then $\underset{\sim}{\mathbf{B}}^\partial = \mathrm{Hom}(\underset{\sim}{\mathbf{B}}, \underset{\sim}{\mathbf{A}}) \in \mathbf{SP}(\mathbb{A})$
- $\underset{\sim}{\mathbf{A}}$ should be algebraic over $\mathbb{A}$

Natural dualities

For $\mathbb{B} \in \mathbf{SP}(\mathbb{A})$ there are mappings

$$\begin{array}{rccl} e_{\mathbb{B}}: & \mathbb{B} & \to & \mathbb{B}^{\partial\partial} \\ & x & \mapsto & \left(e_{\mathbb{B}}(x) : \mathbb{B}^{\partial} \to \underset{\sim}{\mathbf{A}} : y \mapsto y(a)\right), \end{array}$$

$\underset{\sim}{\mathbf{A}}$ **dualizes** $\mathbb{A}$ if $e_{\mathbb{B}}$ are isomorphisms for all $\mathbb{B} \in \mathbf{SP}(\mathbb{A})$.

Examples of naturally dualizable algebras:

- Groups whose Sylow subgroups are abelian [Nickodemus 2007]
- Rings whose Jacobson radical squares to (0) [Clark, Idziak, Sabourin, Szabo, Willard 2001]
- Algebras with a compatible semilattice operation [Davey, Jackson, Pitkethly, Taukder 2007]
- Algebras with near unanimity term [Davey, Heindorf, McKenzie 1995]

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- ~~Naturally dualizable~~
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Omitting $\{\mathbf{1}, \mathbf{5}\}$

TCT type **1** is the unary type, **5** is the semilattice type

If $\mathbb{A}$ finite and $\mathcal{V}(\mathbb{A})$ is congruence modular then $\mathcal{V}(\mathbb{A})$ omits **1** and **5**.

Theorem (Kearnes, Kiss)

The following are equivalent for a variety $\mathcal{V}$.

1. $\mathcal{V}$ *omits types* $\mathbf{1}$ *and* $\mathbf{5}$.
2. $\mathcal{V}$ *has a sequence of idempotent terms* $f_i(x, y, v, u)$ *for* $0 \leq i \leq 2m+1$, *such that*
    1. $\mathcal{V} \models f_0(x, y, u, v) \approx x$ *and* $\mathcal{V} \models f_{2m+1}(x, y, u, v) \approx v$,
    2. $\mathcal{V} \models f_i(x, y, y, y) \approx f_{i+1}(x, y, y, y)$ *for all even* i,
    3. $\mathcal{V} \models f_i(x, x, y, y) \approx f_{i+1}(x, x, y, y)$ *for all odd* i, *and*
    4. $\mathcal{V} \models f_i(x, y, x, y) \approx f_{i+1}(x, y, x, y)$ *for all odd* i.

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Cube terms

For a variety $\mathcal{V}$ the term $t(x_1, \dots, x_n)$ is a **cube term for** $\mathcal{V}$, if for all $1 \leq i \leq n$ there is a choice of $u_1, \dots, u_n \in \{x, y\}$ with $u_i = y$ such that

$$\mathcal{V} \models t(u_1, \dots, u_n) \approx x.$$

Two examples:

- The "weakest" cube term is

$$t\begin{pmatrix} y & x & \cdots & x \\ x & y & \cdots & x \\ \vdots & & \ddots & \vdots \\ x & x & \dots & y \end{pmatrix} \approx \begin{pmatrix} x \\ x \\ \vdots \\ x \end{pmatrix}$$

- The "strongest" cube term "is"

$$t\begin{pmatrix} y & x & x & y & y & x & y \\ x & y & x & y & x & y & y \\ x & x & y & x & y & y & y \end{pmatrix} \approx \begin{pmatrix} x \\ x \\ x \end{pmatrix}$$

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Why?

Theorem (Davey, Heindorf, McKenzie 1995)

The following are equivalent for a finite algebra $\mathbb{A}$.

1. $\mathbb{A}$ *is dualizable and* $\mathcal{V}(\mathbb{A})$ *is congruence distributive.*
2. $\mathbb{A}$ *has a near unanimity term.*

Question

Is there a similar connection between the existence of a cube term for congruence modular $\mathcal{V}(\mathbb{A})$ *and dualizability of* $\mathbb{A}$*?*

Intuition:

- The cube term is a generalization of the NU term.
- A cube term for $\mathcal{V}$ implies CM just like an NU term for $\mathcal{V}$ implies CD.

Counter-evidence:

- The group $\mathbb{S}_3$ is in a CM variety, has a cube term, and is dualizable, but the algebra obtained by adding constants is non-dualizable.

Inherent non-dualizability

A finite algebra $\mathbb{A}$ is **inherently non-dualizable** if for all finite algebras $\mathbb{B}$,

$$\mathbb{A} \in \mathbf{SP}(\mathbb{B}) \qquad \text{implies} \qquad \mathbb{B} \text{ non-dualizable.}$$

Theorem (Davey, Idziak, Lampe, McNulty 1997)

Let $\mathbb{A}$ be a finite algebra, $\mathbb{B} \leq \mathbb{A}^Z$, $B_0 \subseteq B$ be an infinite subset such that

1. *there is a function $\varphi : \omega \to \omega$ such that for all $k \in \omega$ and all $\theta \in Con(\mathbb{B})$ of index at most k, $\theta|_{B_0}$ has a unique block of size greater than $\varphi(k)$; and*
2. *if the element $g \in A^Z$ is defined by $g(z) = \pi(a_z)$ for $z \in Z$, where a_z is an element of the unique block of $\ker(\pi_z)|_{B_0}$ of size greater than $\varphi(|B|)$, then $g \notin B$.*

Then $\mathbb{A}$ is inherently non-dualizable.

Elements of the proof (assuming idempotency)

Pick $\mathbb{A}$ the smallest idempotent dualizable algebra omitting **1** and **5** **without** a cube term. $\mathbb{A}$ has a cube term blocker, so there are $a, b \in A$ such that there is no $t(\cdots)$ with

$$t\left(\text{cube pattern on } (a, b)\right) = \begin{pmatrix} a \\ \vdots \\ a \end{pmatrix}.$$

Define elements of A^ω

$$\alpha_{i_1 \cdots i_n} = \begin{cases} b & \text{if } i \in \{i_1, \ldots, i_n\} \\ a & \text{otherwise.} \end{cases}$$

We will apply the non-dualizability theorem to

$$C_0 = \{\alpha_i \mid i \in \omega\} \qquad \mathbb{C} = \mathrm{Sg}^{\mathbb{A}^\omega}(C_0)$$

Elements of the proof (assuming idempotency)

To show:

1. there is a function $\varphi : \omega \to \omega$ such that for all $k \in \omega$ and all $\theta \in \mathrm{Con}(\mathbb{C})$ of index at most k, $\theta|_{C_0}$ has a unique block of size greater than $\varphi(k)$

Take $\varphi(k) = 1$, $\theta \in \mathrm{Con}(\mathbb{C})$, and assume that there are two $\theta|_{C_0}$-blocks of size > 1:

$$S = \{\alpha_1, \alpha_3, \ldots\} \qquad \text{and} \qquad T = \{\alpha_2, \alpha_4, \ldots\}$$

Claim

The set $\{\alpha_{12}, \alpha_{14}, \alpha_{23}, \alpha_{34}, \alpha_{124}, \alpha_{123}, \alpha_{134}, \alpha_{234}\}$ *lies in single θ-block.*

Proof of claim.

... (use the WNU and cube term blockers) ... ∎

Elements of the proof (assuming idempotency)

To show:

1. there is a function $\varphi : \omega \to \omega$ such that for all $k \in \omega$ and all $\theta \in \mathrm{Con}(\mathbb{C})$ of index at most k, $\theta|_{C_0}$ has a unique block of size greater than $\varphi(k)$

Claim

$\alpha_1 \; \theta \; \alpha_2$.

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Proof of claim.

$\mathbb{A}$ omits **1** and **5**, so we have terms

1. $\mathcal{V} \models f_0(x, y, u, v) \approx x$ and $\mathcal{V} \models f_{2m+1}(x, y, u, v) \approx v$,
2. $\mathcal{V} \models f_i(x, y, y, y) \approx f_{i+1}(x, y, y, y)$ for all even i,
3. $\mathcal{V} \models f_i(x, x, y, y) \approx f_{i+1}(x, x, y, y)$ for all odd i, and
4. $\mathcal{V} \models f_i(x, y, x, y) \approx f_{i+1}(x, y, x, y)$ for all odd i.

If i is even then $f_i(\alpha_1, \alpha_{12}, \alpha_{12}, \alpha_{12}) = f_{i+1}(\alpha_1, \alpha_{12}, \alpha_{12}, \alpha_{12})$.
If i is odd then by the previous claim,

$$f_i(\alpha_1, \alpha_{12}, \alpha_{12}, \alpha_{12}) \; \theta \; f_i(\alpha_1, \alpha_{12}, \alpha_{34}, \alpha_{234})$$

$$= f_i \begin{pmatrix} b & b & a & a \\ a & b & a & b \\ a & a & b & b \\ a & a & b & b \end{pmatrix} = f_{i+1} \begin{pmatrix} b & b & a & a \\ a & b & a & b \\ a & a & b & b \\ a & a & b & b \end{pmatrix}$$

$$= f_{i+1}(\alpha_1, \alpha_{12}, \alpha_{34}, \alpha_{234}) \; \theta \; f_{i+1}(\alpha_1, \alpha_{12}, \alpha_{12}, \alpha_{12}). \quad \bullet$$

Elements of the proof (assuming idempotency)

To show:

2. if the element $g \in A^\omega$ is defined by $g(z) = \pi_z(a_z)$ for $z \in \omega$, where a_z is an element of the unique block of $\ker(\pi_z)|_{C_0}$ of size greater than $\varphi(|C|)$, then $g \notin C$

Observe:

- $\ker(\pi_z)|_{C_0}$ has two blocks: $X_z = \{\alpha_z\}$ and $Y_z = \{\alpha_i \mid i \neq z\}$.
- Therefore Y_z is the unique large block, and $\pi_z(\alpha_i) = a$ for $i \neq z$.
- Thus $g(z) = a$ for all $z \in \omega$.

Elements of the proof (assuming idempotency)

Claim

$g \notin C$

Proof of claim.

Suppose that $g \in C$. Then there is a term $t(\cdots)$ such that $t(\alpha_1, \ldots, \alpha_m) = g$. That is,

$$t \begin{pmatrix} b & a & \cdots & a \\ a & b & \cdots & a \\ \vdots & & \ddots & \vdots \\ a & a & \cdots & b \end{pmatrix} = \begin{pmatrix} a \\ a \\ \vdots \\ a \end{pmatrix}$$

We chose a and b so that this couldn't happen! ∎

By the non-dualizability theorem $\mathbb{A}$ is inherently non-dualizable. This is a contradiction.

Theorem

Let $\mathbb{A}$ be a finite algebra such that $\mathcal{V}(\mathbb{A})$ omits types $\mathbf{1}$ and $\mathbf{5}$. If $\mathbb{A}$ admits a natural duality, then $\mathbb{A}$ has a cube term.

Thank you!