Optimal strong Maltsev conditions for congruence meet-semidistributivity

Matthew Moore

Vanderbilt University

January 7, 2015
BLAST at New Mexico State University

Joint work with...

Jelena Jovanović, Belgrade University

Petar Marković, University of Novi Sad

Ralph McKenzie, Vanderbilt University

Outline

1. SD($\wedge$) and the CSP
2. Known Maltsev conditions
3. Better Maltsev conditions for $SD(\wedge)$
4. Some conjectured Maltsev conditions

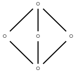
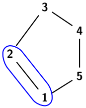
SD(∧) varieties

Definition

A variety $\mathcal{V}$ is **congruence meet-semidistributive** (SD(∧)) if for every algebra $\mathbb{A} \in \mathcal{V}$,

$$\mathrm{Con}(\mathbb{A}) \models \big[(x \wedge y \approx x \wedge z) \to (x \wedge y \approx x \wedge (y \vee z)) \big].$$

(for $\mathcal{V}$ locally finite . . .)

- (⇔)  is not a sublattice of $\mathrm{Con}(\mathbb{A})$ for all $\mathbb{A} \in \mathcal{V}$
- (⇔) $\mathcal{V}$ omits TCT types **1** and **2**: 

- (⇔) $\mathrm{Con}(\mathbb{A}) \models [x, y] \approx x \wedge y$ for any $\mathbb{A} \in \mathcal{V}$ (*congruence neutral*)
- Park's Conjecture is true (if $\mathcal{V}$ has finite residual bound, then $\mathcal{V}$ is finitely based) [Willard 2000]
- CSP($\mathbb{A}$) can be solved using local consistency checking [Barto, Kozik 2014]

Definition

Let $\mathbb{A}$ be a finite algebra. An instance of the **constraint satisfaction problem for** $\mathbb{A}$, written CSP($\mathbb{A}$), is a triple $(V; A; \mathcal{C})$:

- V is a finite nonempty set of variables
- $\mathcal{C}$ is a finite nonempty set of **constraints**
    - for each $C \in \mathcal{C}$ there is $W \subseteq V$ such that $C \leq \mathbb{A}^W$
    - W is called the **scope** of C
    - $|W|$ is called the **arity** of C

An instance of CSP($\mathbb{A}$) is said to have a **solution** if there is an assignment of elements of A to the variables V so that all constraints are true.

Example:

$$\mathbb{A} \models \exists \overline{x} \left[(x_1, x_3) \in R_1 \ \wedge \ x_2 \in R_2 \ \wedge \ x_3 \in R_2 \ \wedge \ (x_2, x_3, x_1) \in R_3 \right]$$

How hard is it to decide if a solution exists?

CSP($\mathbb{A}$)

Theorem (Barto 2014)

If $\mathbb{A}$ is idempotent and $\mathcal{V}(\mathbb{A})$ is SD($\wedge$), then every $(2,3)$-minimal instance of CSP($\mathbb{A}$) has a solution.

Definition

Let $(V; A; \mathcal{C})$ be a CSP instance.

- $(V; A; \mathcal{C})$ is **2-consistent** if for every $U \subseteq V$ with $|U| \leq 2$ and every pair of constraints $C, D \in \mathcal{C}$ containing U in their scopes, $C|_U = D|_U$.
- $(V; A; \mathcal{C})$ is **$(2,3)$-minimal** if it is 2-consistent and every subset $U \leq V$ with $|U| \leq 3$ is contained in the scope of some constraint.

Typical usage: build a $(2,3)$-minimal CSP($\mathbb{F}(\overline{x})$) instance (in l.f. idemp. SD($\wedge$) variety) and use combinatorics.

Theorem

$\mathcal{V}$ is SD($\wedge$) iff $\mathcal{V}$ satisfies an idempotent Maltsev condition which fails in any variety of modules.

Some known Maltsev characterizations

A variety $\mathcal{V}$ is said to satisfy WNU(n) if it has an idempotent n-ary term $t(\cdots)$ such that

$$\mathcal{V} \models t(y, x, \ldots, x) \approx t(x, y, x, \ldots, x) \approx \cdots \approx t(x, \ldots, x, y).$$

This is the weak near unanimity term condition.

TFAE for locally finite $\mathcal{V}$

- $\mathcal{V}$ is SD($\wedge$)
- there exists $n > 1$ such that $\mathcal{V} \models$ WNU(k) for all $k \geq n$ [Maroti, McKenzie 2008]
- $\mathcal{V}$ satisfies WNU(4) via $t(\cdots)$ and WNU(3) via $s(\cdots)$ and

  $$t(y, x, x, x) \approx s(y, x, x)$$

  [Kozik, Krokhin, Valeriote, Willard 2013]

"Better" Maltsev conditions

Let Σ and Ω be Maltsev conditions.
(some sets of equations in some language)

- Write $\Sigma \preceq \Omega$ if any variety which realizes Ω must also realize Σ.
- This induces a preorder.
- If $\Sigma \preceq \Omega$, we say Ω is **stronger** than Σ.
- If $\Sigma \preceq \Omega \preceq \Sigma$, we say the conditions are **equivalent** and write $\Sigma \sim \Omega$.

Many strong Maltsev conditions which are not equivalent are equivalent **within the class** of locally finite varieties.

Characterizations of SD($\wedge$) for locally finite $\mathcal{V}$

$t(\dots), s(\dots)$ WNU's
$t(yxxx) \approx s(yxx)$

$\exists n \forall k > n$ there is k-ary WNU

A restricted $\preceq$-minimal characterization

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there are idempotent terms $p(\cdots)$, $q(\cdots)$ such that

$$p(x,x,y) \approx p(x,y,x) \approx p(y,x,x) \approx q(x,y,x) \text{ and}$$
$$q(x,x,y) \approx q(x,y,y)$$

There is no idempotent strong Maltsev condition characterizing SD($\wedge$) in the language with one ternary and any number of binary operation symbols.

In the class of all strong idempotent Maltsev conditions in a language consisting of 2 ternary operation symbols, a computer search produced as a candidate for being $\preceq$-minimal for characterizing SD($\wedge$) varieties. [Jovanović 2013]

Characterizations of SD($\wedge$) for locally finite $\mathcal{V}$

Language $\supseteq$
two 3-ary ops

$p(xxy) \approx p(xyx)$
$\approx p(yxx) \approx q(xyx)$
$\approx q(xxy) \approx q(xyy)$

$t(\ldots), s(\ldots)$ WNU's
$t(yxxx) \approx s(yxx)$

$\exists n \forall k > n$ there
is k-ary WNU

~~Language $=$~~
~~one 3-ary, any # binary~~

Other optimal Maltsev characterizations

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there is an idempotent term $t(\cdots)$ such that

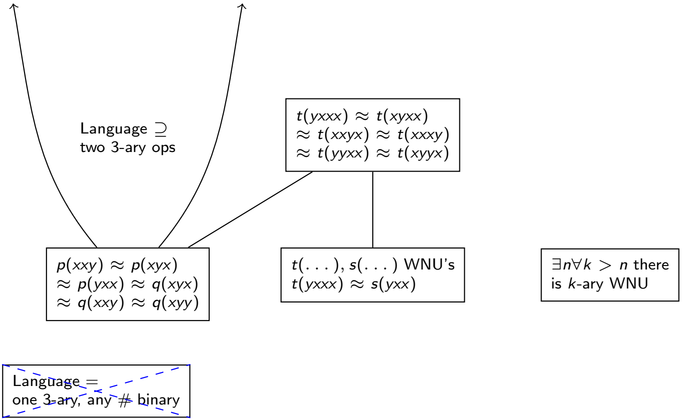
$$\begin{aligned} t(y,x,x,x) &\approx t(x,y,x,x) \approx t(x,x,y,x) \approx t(x,x,x,y) \\ &\approx t(y,y,x,x) \approx t(y,x,y,x) \approx t(x,y,y,x) \end{aligned}$$

Look at the relation

$$U = \mathrm{Sg}\begin{pmatrix} x & x & x & y \\ x & x & y & x \\ x & y & x & x \\ y & x & x & x \\ y & y & x & x \\ y & x & y & x \\ x & y & y & x \end{pmatrix}$$

in $\mathbb{F}^{\mathcal{V}}(x,y)$, plus 11 ternary relations, plus 3 binary. Then use a (difficult) Ramsey-style argument.

Characterizations of SD(∧) for locally finite $\mathcal{V}$



Better WNU's

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there is a term $t(x, y)$ and for all $n \geq 3$,

- *there exists n-ary WNU, $w(\cdots)$ and*
- $t(x, y) = w(y, x, \ldots, x)$.

Proof.

(is there time?) □

Characterizations of SD($\wedge$) for locally finite $\mathcal{V}$

Language $\supseteq$ two 3-ary ops

$t(yxxx) \approx t(xyxx) \approx t(xxyx) \approx t(xxxy) \approx t(yyxx) \approx t(xyyx)$

$\exists t(xy)$ s.t. $\forall n \geq 3$ $\exists n$-ary WNU $w(\ldots)$ $w(x \ldots xy) \approx t(xy)$

$p(xxy) \approx p(xyx) \approx p(yxx) \approx q(xyx) \approx q(xxy) \approx q(xyy)$

$t(\ldots), s(\ldots)$ WNU's $t(yxxx) \approx s(yxx)$

$\exists n \forall k > n$ there is k-ary WNU

~~Language = one 3-ary, any # binary~~

How much better can we do?

Theorem

Any strong Maltsev condition of the form

$$f(x,\dots,x) \approx x \qquad \text{and} \qquad f(y_1,\dots,y_n) \approx f(z_1,\dots,z_n),$$

where $y_i, z_j \in \{x_1,\dots,x_m\}$, that is realized in a nontrivial semilattice can also be realized in a nontrivial module.

Characterizations of SD($\wedge$) for locally finite $\mathcal{V}$

Language $\supseteq$
two 3-ary ops

$t(yxxx) \approx t(xyxx)$
$\approx t(xxyx) \approx t(xxxy)$
$\approx t(yyxx) \approx t(xyyx)$

$\exists t(xy)$ s.t. $\forall n \geq 3$
$\exists n$-ary WNU $w(\ldots)$
$w(x \ldots xy) \approx t(xy)$

$p(xxy) \approx p(xyx)$
$\approx p(yxx) \approx q(xyx)$
$\approx q(xxy) \approx q(xyy)$

$t(\ldots), s(\ldots)$ WNU's
$t(yxxx) \approx s(yxx)$

$\exists n \forall k > n$ there
is k-ary WNU

~~Language $=$~~
~~one 3-ary, any # binary~~

~~Language $=$~~
~~idempotent f, $f(\overline{x}) = f(\overline{y})$~~

Candidates for "least-equations"-optimal

Amongst all idempotent strong Maltsev conditions of the form

$$f(\overline{x}) \approx f(\overline{y}) \approx f(\overline{z}),$$

for $f(\cdots)$ of arity ≤ 4, a computer search eliminates all but two candidates:

$$t\begin{pmatrix} x & x & y & z \\ y & z & y & x \\ x & z & z & y \end{pmatrix} = \begin{pmatrix} w \\ w \\ w \end{pmatrix} \qquad t\begin{pmatrix} x & x & y & z \\ y & x & z & x \\ y & z & x & y \end{pmatrix} = \begin{pmatrix} w \\ w \\ w \end{pmatrix}$$

Problem

Prove that a locally finite SD($\wedge$) variety satisfies one (or both) of the Maltsev conditions above.

Characterizations of SD($\wedge$) for locally finite $\mathcal{V}$

???

$t(xxyz) \approx t(yzyx) \approx t(xzzy)$

$t(xxyz) \approx t(yxzx) \approx t(yzxy)$

???

Language $\supseteq$ two 3-ary ops

$t(yxxx) \approx t(xyxx) \approx t(xxyx) \approx t(xxxy) \approx t(yyxx) \approx t(xyyx)$

$\exists t(xy)$ s.t. $\forall n \geq 3$ $\exists n$-ary WNU $w(\dots)$ $w(x \dots xy) \approx t(xy)$

$p(xxy) \approx p(xyx) \approx p(yxx) \approx q(xyx) \approx q(xxy) \approx q(xyy)$

$t(\dots), s(\dots)$ WNU's $t(yxxx) \approx s(yxx)$

$\exists n \forall k > n$ there is k-ary WNU

~~Language $=$ one 3-ary, any $\#$ binary~~

~~Language $=$ idempotent f, $f(\overline{x}) = f(\overline{y})$~~

WNU's (special and otherwise)

Theorem (JMMM)

A locally finite variety $\mathcal{V}$ is SD($\wedge$) iff there is a term $t(x, y)$ and for all $n \geq 3$,

- *there exists n-ary WNU, $w(\cdots)$ and*
- $t(x, y) = w(y, x, \ldots, x)$.

A WNU $w(\cdots)$ is called **special** if $t(x, t(x, y)) = t(x, y)$ for $t(x, y) = w(y, x, \ldots, x)$.

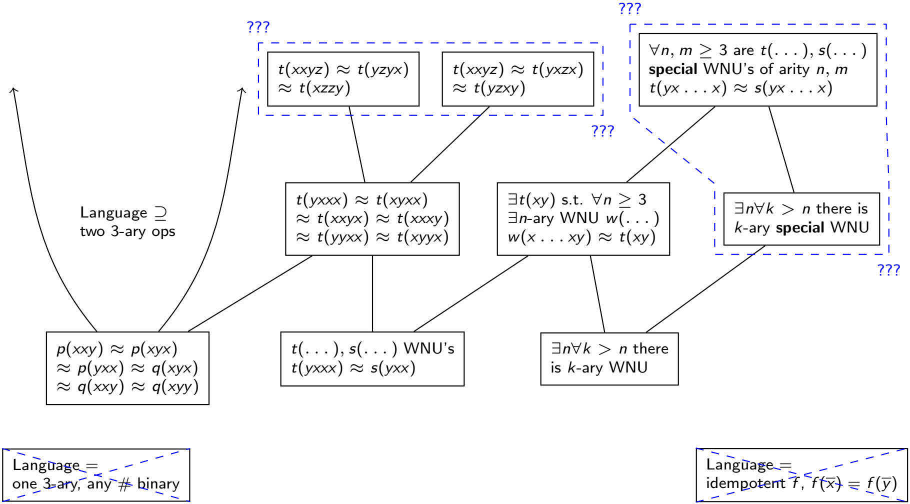
Problem

Prove that the WNU's in the above theorem can be taken to be special.

Problem

A locally finite variety $\mathcal{V}$ is SD($\wedge$) if there exists n such that $\mathcal{V}$ has special WNU's of all arities $k > n$.

Characterizations of SD(∧) for locally finite $\mathcal{V}$



Thank you.

Shanks workshop: Open Problems in Universal Algebra

Vanderbilt University

May 28 – June 1, 2015

`www.math.vanderbilt.edu/~moorm10/shanks/`