Absorption and directed Jónsson terms

Matthew Moore

Vanderbilt University

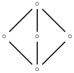
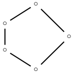
June 11, 2015

This is joint work with

- Alexandr Kazda, Institute of Science and Technology
- Marcin Kozik, Jagiellonian University
- Ralph McKenzie, Vanderbilt University

Congruence distributive algebras

$\mathcal{V}$ is **congruence distributive (CD)** if for every $\mathbb{A} \in \mathcal{V}$ the lattice of congruences of $\mathbb{A}$ is a distributive lattice.

$\Leftrightarrow$ no  or  in congruence lattices

$\Leftrightarrow$ there are **Jónsson terms**: $J_0, \ldots, J_{2n+1}$ such that $\mathcal{V}$ models

$$J_1(x,x,y) = x, \qquad J_{2n+1}(x,y,y) = y, \qquad J_i(x,y,x) = x$$
$$J_{2i+1}(x,y,y) = J_{2i+2}(x,y,y), \qquad J_{2i}(x,x,y) = J_{2i+1}(x,x,y).$$

& locally finite $\Rightarrow$ $\mathcal{V}$ has finite residual bound and is finitely axiomatizable.

Jónsson terms

$$J_1(x,x,y)=x, \qquad J_{2n+1}(x,y,y)=y, \qquad J_i(x,y,x)=x$$
$$J_{2i+1}(x,y,y)=J_{2i+2}(x,y,y), \qquad J_{2i}(x,x,y)=J_{2i+1}(x,x,y).$$

If $J_i(x,x,y) \to J_i(x,y,y)$, then

$$x = J_1(x,x,y) \qquad \begin{aligned} a_2 &= J_2(x,x,y) \\ &= J_3(x,x,y) \end{aligned} \qquad \begin{aligned} a_{2n} &= J_{2n}(x,x,y) \\ &= J_{2n+1}(x,x,y) \end{aligned}$$

$$\begin{tikzcd}
x \arrow[dr] & & a_2 \arrow[dl] \arrow[dr] & & {} \arrow[dl] & \cdots & a_{2n} \arrow[dl] \arrow[dr] & \\
& a_1 & & a_3 & & {} & & y
\end{tikzcd}$$

$$\begin{aligned} a_1 &= J_1(x,y,y) \\ &= J_2(x,y,y) \end{aligned} \qquad \begin{aligned} a_3 &= J_3(x,y,y) \\ &= J_4(x,y,y) \end{aligned} \qquad y = J_{2n+1}(x,y,y)$$

Absorption

A **near unanimity (NU) term** for a variety $\mathcal{V}$ is an idempotent term $t(\cdots)$ such that

$$\mathcal{V} \models t(y, x, \ldots, x) \approx t(x, y, \ldots, x) \approx t(x, \ldots, x, y) \approx x.$$

$$\begin{aligned} \exists \text{ NU term} &\Leftrightarrow \text{CD, every finite algebra is finitely related} \\ &\Leftrightarrow \text{CD, finite algebras are dualizable} \end{aligned}$$

Definition

For algebra $\mathbb{A}$ with subalgebra $\mathbb{B} \leq \mathbb{A}$, $\mathbb{B}$ **absorbs** $\mathbb{A}$ (written $\mathbb{B} \trianglelefteq \mathbb{A}$) if there is an idempotent term $t(\cdots)$ such that for all i,

$$t(B, \ldots, B, \overset{i}{\widehat{A}}, B, \ldots, B) \subseteq B$$

Absorption theorems

Theorem (Barto, Kozik 2012)

Let $\mathcal{V}$ be a locally finite idempotent variety. TFAE

- *$\mathcal{V}$ is a Taylor variety,*
- *for all finite $\mathbb{A}, \mathbb{B} \in \mathcal{V}$ and all linked $\mathbb{R} \leq_{sd} \mathbb{A} \times \mathbb{B}$,*
    - *$\mathbb{R} = \mathbb{A} \times \mathbb{B}$, or*
    - *$\mathbb{A}$ has a proper absorbing subuniverse, or*
    - *$\mathbb{B}$ has a proper absorbing subuniverse.*

Theorem (Barto, Kozik, Willard 2012)

Let $\mathbb{A}$ be a finite algebra and $\mathbb{R} \trianglelefteq \mathbb{S} \leq \mathbb{A}^n$ such that $\mathbb{R} \leq_{sd} \mathbb{A}^n$ and $\Delta_n \leq \mathbb{S}$. Then $\Delta_n \cap R \neq \emptyset$.

From near unanimity to directed Jónsson terms

Let $t(\cdots)$ be an n-ary near unanimity term for $\mathbb{A}$:

$$\mathbb{A} \models t(y, x, \ldots, x) \approx t(x, y, \ldots) \approx t(x, \ldots, x, y) \approx x.$$

Define terms $D_1, \ldots, D_n$ by

$$D_1(x, y, z) = t(x, \ldots, x, y), \qquad D_n(x, y, z) = t(y, z, \ldots, z),$$
$$D_i(x, y, z) = t(x, \ldots, x, y, z, \ldots, z) \qquad y \text{ in the } (n - i + 1)th \text{ place.}$$

It's not hard to see that these satisfy the **directed Jónsson identities**:

$$D_1(x, x, y) = x, \qquad D_n(x, y, y) = y,$$
$$D_i(x, y, y) = D_{i+1}(x, x, y), \qquad D_i(x, y, x) = x.$$

These identities imply CD.

$$\begin{tikzcd}
 & & \text{Jónsson} \arrow[d, Leftrightarrow] \\
\text{NU} \arrow[r, Rightarrow] \arrow[rr, Leftrightarrow, bend right] & \text{dir. Jónsson} \arrow[r, Rightarrow] & \text{CD} \\
 & & \text{CD, fin. related}
\end{tikzcd}$$

Directed Jónsson terms

$$D_1(x, x, y) = x, \qquad D_n(x, y, y) = y,$$
$$D_i(x, y, y) = D_{i+1}(x, x, y), \qquad D_i(x, y, x) = x$$

If $D_i(x, x, y) \to D_i(x, y, y)$, then

$$x = D_1(x, x, y)$$
$$a_2 = D_2(x, y, y) = D_3(x, x, y)$$
$$a_n = D_n(x, y, y) = D_{n+1}(x, x, y)$$

$$\begin{tikzcd}
x \arrow[dr] & & a_2 \arrow[dr] & & \cdots & & a_n \arrow[dr] & \\
& a_1 \arrow[ur] & & a_3 \arrow[ur] & & \arrow[ur] & & y
\end{tikzcd}$$

$$a_1 = D_1(x, y, y) = D_2(x, x, y)$$
$$a_3 = D_3(x, y, y) = D_4(x, x, y)$$
$$y = D_n(x, y, y)$$

From absorption to weak directed Jónsson terms

Suppose that $\mathbb{B} \trianglelefteq \mathbb{A}$ via the *n*-ary term $t(\cdots)$ and define

$$D_1(x,y,z) = t(x,\dots,x,y), \qquad D_n(x,y,z) = t(y,z,\dots,z),$$
$$D_i(x,y,z) = t(x,\dots,x,y,z,\dots,z) \qquad y \text{ in the } (n-i+1)th \text{ place.}$$

These satisfy the **weak directed Jónsson identities**:

$$D_1(x,x,y) = x, \qquad D_n(x,y,y) = y,$$
$$D_i(x,y,y) = D_{i+1}(x,x,y), \qquad \xcancel{D_i(x,y,x) = x},$$

Jónsson absorption

$\mathbb{B} \trianglelefteq \mathbb{A}$ via $t(\cdots)$, from which the D_i were defined.

$$D_1(x,x,y) = x, \qquad D_n(x,y,y) = y,$$
$$D_i(x,y,y) = D_{i+1}(x,x,y), \qquad D_i(B,A,B) \subseteq B,$$

If we have terms D_i and $\mathbb{B} \leq \mathbb{A}$ satisfying these, we say $\mathbb{B}$ **directed Jónsson absorbs** $\mathbb{A}$ and write $\mathbb{B} \triangleleft_{DJ} \mathbb{A}$.

Similarly, replacing $J_i(x,y,x) = x$ with $J_i(B,A,B) \subseteq B$ in the Jónsson identities, we obtain **Jónsson absorption**, written $\mathbb{B} \triangleleft_J \mathbb{A}$.

$\mathbb{A}$ has Jónsson terms if and only if $\{a\} \triangleleft_J \mathbb{A}$ for all $a \in A$.

$\mathbb{A}$ has directed Jónsson terms if and only if $\{a\} \triangleleft_{DJ} \mathbb{A}$ for all $a \in A$.

Directed Jónsson terms in a finite algebra

Theorem

Suppose that $\mathbb{A}$ is **finite** *and $\mathbb{E}, \mathbb{F} \leq \mathbb{A}^2$ are preorders (i.e. reflexive and transitive). If $\mathbb{E} \lhd_J \mathbb{F}$ then $\mathbb{E} = \mathbb{F}$.*

How to get directed Jónsson terms (finite $\mathbb{A}$): Suppose $\mathcal{V}$ is CD.

- $\mathbb{A}$ is finite, so $\mathcal{V}(\mathbb{A})$ is locally finite.
- $\mathbb{F}_2 = \mathbb{F}(x, z)$ and $\mathbb{F}_3 = \mathbb{F}(x, y, z)$ are finite.
- Let $\mathcal{G} = \{t(x, y, z) \in \mathbb{F}_3 \mid t(x, z, x) = x\}$,
  $F = \big\{\big(t(x, x, z), t(x, z, z)\big) \mid t \in \mathbb{F}_3\big\}$,
  $E = \big\{\big(t(x, x, z), t(x, z, z)\big) \mid t \in \mathcal{G}\big\}$.
- Let $\rightarrow$ and $\dashrightarrow$ be the transitive closures of E and F, resp.
- E, F, $\rightarrow$, and $\dashrightarrow$ are subalgebras of $\mathbb{F}_2$ and $\rightarrow$, $\dashrightarrow$ are preorders.
- The Jónsson terms for $\mathcal{V}$ witness $\mathbb{E} \lhd_J \mathbb{F}$, so $\rightarrow \lhd_J \dashrightarrow$, so $\rightarrow = \dashrightarrow$.
- $(x, z) \in F$, so $x \rightarrow z$, so there are witnessing E-related terms $D_i \in \mathcal{G}$.
- These are the directed Jónsson terms!

$$
\begin{aligned}
\mathcal{G} &= \big\{ t(x,y,z) \in \mathbb{F}_3 \mid t(x,z,x) = x \big\}, \\
E &= \big\{ \big(t(x,x,z), t(x,z,z) \big) \mid t \in \mathcal{G} \big\}, \\
\to &= \text{transitive closure of } E
\end{aligned}
$$

$x \to z$

There is a sequence in E witnessing $x \to z$:

$$
\begin{array}{ccccc}
x & (D_2(x,x,z), D_2(x,z,z)) & & \cdots & (D_n(x,x,z), D_n(x,z,z)) \\
\| & \| \qquad\qquad \| & & & \| \\
(D_1(x,x,z), D_1(x,z,z)) & & (D_3(x,x,z), D_3(x,z,z)) & & z
\end{array}
$$

Every pair comes from $\mathcal{G}$, so $D_i(x,z,x) = x$.

These are the directed Jónsson terms!

Directed Jónsson terms in general

Theorem

Let $\mathcal{V}$ be a variety and $\mathcal{J}$ a system of weak Jónsson terms. Then there is a system of weak directed Jónsson terms $\mathcal{D}$ such that for all $\mathbb{A}, \mathbb{B} \in \mathcal{V}$ if $\mathbb{B} \triangleleft_J \mathbb{A}$ via $\mathcal{J}$ then $\mathbb{B} \triangleleft_{DJ} \mathbb{A}$ via $\mathcal{D}$.

How to get directed Jónsson terms:

- Let $\mathcal{V}$ be a variety with Jónsson terms $\mathcal{J}$ and let $\mathbb{F}_3^{id}$ be the idempotent reduct of $\mathbb{F}(x, y, z)$.
- $\mathcal{J}$ are Jónsson terms for $\mathcal{V}$, so $\{x\} \triangleleft_J \mathbb{F}_3^{id}$.
- Thus there are directed Jónsson terms $\mathcal{D}$ such that $\{x\} \triangleleft_{DJ} \mathbb{F}_3^{id}$.
- The elements of $\mathcal{D}$ are weak directed Jónsson terms plus the identities $D_i(x, y, x) = x$.
- Thus $\mathbb{F}_3^{id}$ has directed Jónsson terms.

Jónsson if and only if directed Jónsson

Theorem

Let $\mathcal{V}$ be any variety. $\mathcal{V}$ is congruence distributive if and only if it has directed Jónsson terms.

$$
\begin{tikzcd}
 & & \text{Jónsson} \arrow[dl, Rightarrow, "\text{for all } \mathcal{V}"'] \arrow[d, Leftrightarrow] \\
\text{NU} \arrow[r, Rightarrow] & \text{dir. Jónsson} \arrow[r, Rightarrow] & \text{CD}
\end{tikzcd}
$$

New Maltsev conditions

We can prove similar absorption theorems for (directed) Gumm terms and Pixley terms.

Theorem

Let $\mathcal{V}$ be any variety.

- *$\mathcal{V}$ is congruence distributive iff it has directed Jónsson terms.*
- *For any integer $k \geq 1$, a $\mathcal{V}$ is congruence distributive and has $(k+1)$-permuting congruences iff it has Pixley terms of length k.*
- *$\mathcal{V}$ is congruence modular iff it has directed Gumm terms.*

Thank you.