Relational coloring of varieties containing a cube term

Matthew Moore

Vanderbilt University

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Coloring varieties containing a cube term

Cube terms

An n-**dimensional cube term** for an algebra $\mathbb{A}$ is a term $t(\cdots)$ such that $\mathbb{A} \models t(u_1, \ldots, u_m) \approx \overline{x}$ for some $u_i \in \{x, y\}^n \setminus \{x\}^n$.

Example

$$\mathbb{A} \models t \begin{pmatrix} y & x & x & y & y & x & y \\ x & y & x & y & x & y & y \\ x & x & y & x & y & y & y \end{pmatrix} \approx \begin{pmatrix} x \\ x \\ x \end{pmatrix}$$

Finite idempotent $\mathbb{A}$ has a cube term...

$\Leftrightarrow$ $\mathbb{A}$ has few subpowers: $\log_2 |\mathbf{S}(\mathbb{A}^n)| \in \mathcal{O}(n^k)$.

$\Leftrightarrow$ $\mathbb{A}$ is congruence modular and finitely related.

$\Leftrightarrow$ $\mathbb{A}$ has no cube term blockers: $\mathbb{D} < \mathbb{C} \leq \mathbb{A}$ with $C^n \setminus (C \setminus D)^n \leq \mathbb{A}^n$.

$\Rightarrow$ CSP($\mathbb{A}$) is tractable.

Coloring

Let $\mathcal{A}$ be a clone and $\mathbb{B} = \big(B; (R_j)_{j\in J}\big)$ be a relational structure.

Let $F^{\mathcal{A}}(B)$ be be the free algebra in $\mathcal{V}(\langle A; \mathcal{A}\rangle)$ with generators B.

For $R_i \in (R_j)_{j\in J}$ let

$$R_i^{\mathcal{A}} = \text{closure of } \big\{(b_1, \dots, b_n) \in F^{\mathcal{A}}(B)^n \mid (b_1, \dots, b_n) \in R_i\big\} \text{ under } \mathcal{A}$$

Let $\mathbb{F}^{\mathcal{A}}(\mathbb{B}) = \big(F^{\mathcal{A}}(B); (R_j^{\mathcal{A}})_{j\in J}\big)$.

A **coloring** of $\mathcal{A}$ by $\mathbb{B}$ is a relational homomorphism

$$c : \mathbb{F}^{\mathcal{A}}(\mathbb{B}) \to \mathbb{B} \qquad \text{such that} \qquad c(b) = b.$$

We say that $\mathcal{A}$ is $\mathbb{B}$-**colorable**. We say that $\mathcal{V}$ is $\mathbb{B}$-**colorable** if $\mathrm{Clo}(\mathcal{V})$ is.

What is it good for?

Theorem (Sequeira)

$\mathcal{V}$ is congruence k-permutable for some k iff $\mathcal{V}$ is not $(\{0,1\};\leq)$-colorable.

Theorem (Sequeira)

Let $D = \{1,2,3,4\}$, $\alpha = 12|34$, $\beta = 13|24$, $\gamma = 12|3|4$, and $\mathbb{D} = (D;\alpha,\beta,\gamma)$. $\mathcal{V}$ is congruence modular iff $\mathcal{V}$ is not $\mathbb{D}$-colorable.

NB: Sequeira calls this “compatibility with projections”.

Colorability results

Theorem

Let $\emptyset \neq C \subsetneq B \subseteq A$ be sets, $R_n = B^n \setminus (B \setminus C)^n$, and $\mathbb{A} = \big(A; (R_n)_{n\in\omega}\big)$.

Idempotent $\mathcal{V}$ has a cube term iff $\mathcal{V}$ is not $\mathbb{A}$-colorable.

Theorem

Let $R_n = \left(\omega^2\right)^n \setminus \left(\Delta_2\right)^n$ and $\mathbb{W} = \big(\omega; (R_n)_{n\in\omega}\big)$.

$\mathcal{V}$ has a weak Taylor term iff $\mathcal{V}$ is not $\mathbb{W}$-colorable.

Theorem

Let $\emptyset \neq C \subsetneq B \subseteq A$ be sets, $R_n = B^n \setminus (B \setminus C)^n$, and $\mathbb{A} = \big(A; (R_n)_{n\in\omega}\big)$.

Idempotent $\mathcal{V}$ has a cube term iff $\mathcal{V}$ is not $\mathbb{A}$-colorable.

Proof.

($\Rightarrow$): Suppose $\mathcal{V}$ has an n-dimensional cube term and is $\mathbb{A}$-colorable.

There is some identity $\mathcal{V} \models t(u_1, \ldots, u_n) \approx \overline{x}$ for some $u_i \in \{x, y\}^n \setminus \{x\}^n$.

Let $c \in C$ and $b \in B \setminus C$.

Substitute b for x and c for y to get $t(u_1, \ldots, u_n) = \overline{b}$ and $u_i \in R_n^{\mathcal{V}}$.

Thus $\overline{b} \in R_n^{\mathcal{V}}$. Then $c(\overline{b}) = \overline{b} \in R_n \subseteq A^n$, a contradiction.

Proof (cont.).

($\Leftarrow$): Suppose $\mathcal{V}$ does not have a cube term.

Write $g \prec_t h$ if we have $u_i \in \{g, h\}^n \setminus \{g\}^n$ such that $t(u_1, \ldots, u_n) = \overline{g}$.

Fix some $c_0 \in C$ and define $c : \mathbb{F}^{\mathrm{Clo}(\mathcal{V})}(A) \to \mathbb{A}$ by

$$c(f) = \begin{cases} a & \text{if } f = a \in A \subseteq F^{\mathrm{Clo}(\mathcal{V})}(A), \\ b & \text{else if } \exists b \in B, t(\cdots) \text{ such that } b \prec_t f, \\ c_0 & \text{else.} \end{cases}$$

Clearly $c(a) = a$ for $a \in A$. Does c preserve relations?

Proof (cont.).

$$c(f) = \begin{cases} a & \text{if } f = a \in A \subseteq F^{\mathrm{Clo}(\mathcal{V})}(A), \\ b & \text{else if } \exists b \in B, t(\cdots) \text{ such that } b \prec_t f, \\ c_0 & \text{else.} \end{cases}$$

Suppose that c fails to preserve $R_n^{\mathrm{Clo}(\mathcal{V})}$. Then there exists

- $(f_1, \ldots, f_n) \in R_n^{\mathrm{Clo}(\mathcal{V})}$,
- a term $s(\cdots)$,
- and $u_1, \ldots, u_k \in B^n \setminus (B \setminus C)^n \subseteq F^{\mathrm{Clo}(\mathcal{V})}(A)$,

such that

- $s(u_1, \ldots, u_k) = (f_1, \ldots, f_n)$
- and $c(f_i) = b_i \in (B \setminus C)$.

Thus $b_i \prec_{t_i} f_i$. Let $t = t_1 * t_2 * \cdots * t_n * s$.

Then $\exists v_1, \ldots, v_m \in B^n \setminus (B \setminus C)^n$ such that $t(v_1, \ldots, v_m) = (b_1, \ldots, b_n)$.

This is a free algebra, so substitute x for $B \setminus C$ and y for C to obtain a cube identity for $t(\cdots)$, a contradiction. $\square$

A corollary

Theorem

Let $\emptyset \neq C \subsetneq B \subseteq A$ *be sets,* $R_n = B^n \setminus (B \setminus C)^n$*, and* $\mathbb{A} = \big(A; (R_n)_{n\in\omega}\big)$.

Idempotent $\mathcal{V}$ *has a cube term iff* $\mathcal{V}$ *is not* $\mathbb{A}$*-colorable.*

Let $A = B = \{0,1\}$ and $C = \{0\}$.

Then the polymorphism clone of $\mathbb{A}$ is generated by $\hat{\mathbb{2}} = \langle \{0,1\}; \not\rightarrow \rangle$:
$\mathrm{Pol}(\mathbb{A}) = \mathrm{Clo}(\hat{\mathbb{2}})$.

Corollary

Idempotent $\mathcal{V}$ *has a cube term iff* $\mathcal{V}$ *is not* $\mathrm{Rel}(\hat{\mathbb{2}})$*-colorable.*

Clone homomorphisms

Definition

A homomorphism between clones $\mathcal{A}$ and $\mathcal{B}$ is an arity-preserving mapping $\varphi : \mathcal{A} \to \mathcal{B}$ such that

$$\varphi(\pi_k) = \pi_k \qquad \text{and} \qquad \varphi(f(g_1, \dots, g_n)) = \varphi(f)(\varphi(g_1), \dots, \varphi(g_n)).$$

An $\mathcal{L}$-homomorphism between clones $\mathcal{A}$ and $\mathcal{B}$ is an arity preserving mapping $\varphi : \mathcal{A} \to \mathcal{B}$ such that

$$\varphi(\pi_k) = \pi_k \qquad \text{and} \qquad \varphi(f(\pi_{i_1}, \dots, \pi_{i_n})) = \varphi(f)(\pi_{i_1}, \dots, \pi_{i_n}).$$

Let $\mathcal{A}$ and $\mathcal{B}$ be clones, and let $\varphi : \mathcal{A} \to \mathcal{B}$ be an arity preserving map.

Let $\mathbb{A} = (A; \mathcal{A})$, and $\mathbb{B} = (B; \mathcal{B})$.

φ is a clone homomorphism iff $f \approx g$ in $\mathbb{A}$ implies $\varphi(f) \approx \varphi(g)$ in $\mathbb{B}$.

φ is a clone $\mathcal{L}$-homomorphism iff every identity $f \approx g$ in $\mathbb{A}$ not involving composition implies $\varphi(f) \approx \varphi(g)$ in $\mathbb{B}$.

Example

Consider $\mathbb{2} = \langle \{0,1\}; \to \rangle$ and $\hat{\mathbb{2}} = \langle \{0,1\}; \not\to \rangle$.

No clone homomorphisms between $\mathrm{Clo}(\hat{\mathbb{2}})$ and $\mathrm{Clo}(\mathbb{2})$ since $\to$ cannot be defined as a term in $\not\to$ and $\not\to$ cannot be defined as a term in $\to$.

This is also witnessed by the identities

$$\mathbb{2} \models y \to (x \to x) \approx (x \to x), \qquad \hat{\mathbb{2}} \models (x \not\to x) \not\to y \approx (x \not\to x).$$

Given a function $f : \{0,1\}^n \to \{0,1\}$ define
$\delta(f(x_1, \ldots, x_n)) = \neg f(\neg x_1, \ldots, \neg x_n)$, where $\neg$ is boolean negation.

$\delta : \mathrm{Clo}(\mathbb{2}) \to \mathrm{Clo}(\hat{\mathbb{2}})$ is a clone $\mathcal{L}$-homomorphism, but not a clone homomorphism.

Let $s(x,y,z) = x \to (y \to z)$ and $s'(x,y,z) = (x \not\to y) \not\to z$.

$\delta(s) = s'$ and $\delta(s') = s$.

δ translates the identities witnessing the non-existence of a clone homomorphism.

Connection to coloring

Theorem (Barto, Oprsal, Pinsker)

Let $\mathcal{A}$ be a clone and $\mathbb{B}$ a relational structure with polymorphism clone $\mathcal{B}$.

$\mathcal{A}$ is $\mathrm{Rel}(\mathcal{B})$*-colorable iff there is a clone $\mathcal{L}$-homomorphism $\mathcal{A} \to \mathcal{B}$.*

Corollary

Let $\mathcal{V}$ be an idempotent variety. The following are equivalent.

- *$\mathcal{V}$ does not have a cube term,*
- *there is a clone $\mathcal{L}$-homomorphism* $\mathrm{Clo}(\mathcal{V}) \to \mathrm{Clo}(\hat{\mathbb{2}})$.

Interpretability

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties and write $\mathcal{V} \leq \mathcal{W}$ if there is a clone homomorphism $\mathsf{Clo}(\mathcal{V}) \to \mathsf{Clo}(\mathcal{W})$. Say that $\mathcal{V}$ **is interpretable into** $\mathcal{W}$.

Define an equivalence relation by $\mathcal{V} \equiv \mathcal{W}$ if $\mathcal{V} \leq \mathcal{W} \leq \mathcal{V}$.

Modulo this equivalence relation, under the $\leq$ order the class of all varieties is a lattice. This is the **lattice of interpretability types**.

Write $\mathcal{V} \leq_{\mathcal{L}} \mathcal{W}$ if there is a clone $\mathcal{L}$-homomorphism $\mathsf{Clo}(\mathcal{V}) \to \mathsf{Clo}(\mathcal{W})$. Say that $\mathcal{V}$ **is $\mathcal{L}$-interpretable into** $\mathcal{W}$.

Define an equivalence relation by $\mathcal{V} \equiv_{\mathcal{L}} \mathcal{W}$ if $\mathcal{V} \leq_{\mathcal{L}} \mathcal{W} \leq_{\mathcal{L}} \mathcal{V}$.

Example

$\mathcal{V}(\mathbb{2}) \equiv_{\mathcal{L}} \mathcal{V}(\hat{\mathbb{2}})$

$\mathcal{V}_{\mathcal{L}}$

Let $\mathcal{V}$ be a variety with signature σ.

Replace σ by $\mathrm{Clo}(\mathcal{V})$, call the resulting variety $\mathcal{V}_c$ so that $\mathcal{V}_c \equiv \mathcal{V}$.

Let $\Sigma \subseteq \mathrm{Th}(\mathcal{V}_c)$ be the subset of all identities not involving composition.

Let $\mathcal{V}_{\mathcal{L}} = \mathrm{Mod}(\Sigma)$.

Example

If $\mathcal{V}$ has an identity $t(s(x,y),x,y) \approx s(x,y)$, then $\mathcal{V}_c$ will have a new operation symbol $f(x,y,z,w)$ and $\mathrm{Th}(\mathcal{V}_c)$ will have identities

- $t(s(x,y),x,y) \approx s(x,y)$,
- $f(x,y,z,w) \approx t(s(x,y),z,w)$,
- $f(x,y,x,y) \approx s(x,y)$.

Out of these, Σ will contain **only** the last one.

Example

$(x \not\to x) \not\to y \approx (x \not\to x)$ in $\hat{\mathbb{2}}$ becomes $s(x,x,y) \approx x \not\to x$ in $\mathcal{V}(\hat{\mathbb{2}})_{\mathcal{L}}$.

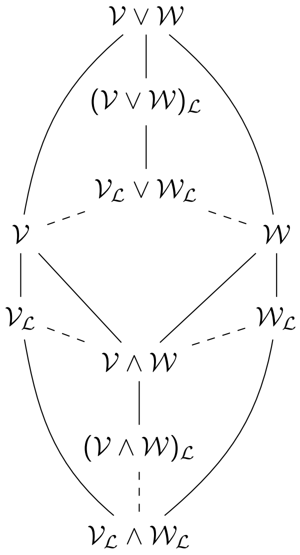
Properties of $\mathcal{V}_{\mathcal{L}}$

Theorem

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties. Then

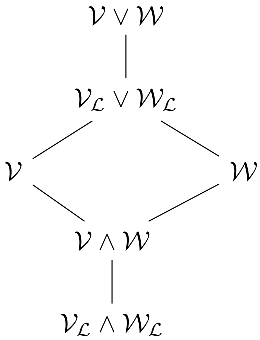
- $\mathcal{V} \equiv_{\mathcal{L}} \mathcal{V}_{\mathcal{L}}$,
- $\mathcal{V}_{\mathcal{L}} \leq \mathcal{W}_{\mathcal{L}}$ *iff* $\mathcal{V} \leq_{\mathcal{L}} \mathcal{W}$,
- *if* $\mathcal{V} \leq \mathcal{W}$ *then* $\mathcal{V}_{\mathcal{L}} \leq \mathcal{W}_{\mathcal{L}}$.
- $\mathcal{V}$ *is defined by linear identities iff* $\mathcal{V} \equiv \mathcal{V}_{\mathcal{L}}$.

Usual interpretability lattice:



Properties of $\mathcal{V}_\mathcal{L}$

$\mathcal{L}$-interpretability lattice:



Theorem

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties. Then

- $\mathcal{V} \equiv_\mathcal{L} \mathcal{V}_\mathcal{L}$,
- $\mathcal{V}_\mathcal{L} \leq \mathcal{W}_\mathcal{L}$ *iff* $\mathcal{V} \leq_\mathcal{L} \mathcal{W}$,
- *if* $\mathcal{V} \leq \mathcal{W}$ *then* $\mathcal{V}_\mathcal{L} \leq \mathcal{W}_\mathcal{L}$.
- $\mathcal{V}$ *is defined by linear identities iff* $\mathcal{V} \equiv \mathcal{V}_\mathcal{L}$.

Corollary

Let $\mathcal{V}$ be an idempotent variety. The following are equivalent.

- *$\mathcal{V}$ does not have a cube term,*
- *$\mathcal{V} \leq_{\mathcal{L}} \mathcal{V}(\hat{\mathbb{2}})$ in the $\mathcal{L}$-interpretability lattice.*
- *$\mathcal{V}_{\mathcal{L}} \leq \mathcal{V}(\hat{\mathbb{2}})_{\mathcal{L}}$ in the interpretability lattice.*

Theorem (Sequeira; Barto, Oprsal, Pinsker)

If $\mathcal{V}$ and $\mathcal{W}$ are defined by linear identities and are $\mathbb{A}$-colorable, then $\mathcal{V} \vee \mathcal{W}$ is $\mathbb{A}$-colorable as well.

Corollary

If idempotent $\mathcal{V}$ and $\mathcal{W}$ are defined by linear identities and $\mathcal{V} \vee \mathcal{W}$ has a cube term, then one of $\mathcal{V}$ or $\mathcal{W}$ does.

Question

Is having a cube term join-prime in the idempotent interpretability lattice?

Answer is tentatively "yes".

Question

Is there a characterization of when $(\mathcal{V} \vee \mathcal{W})_{\mathcal{L}} \equiv \mathcal{V}_{\mathcal{L}} \vee \mathcal{W}_{\mathcal{L}}$*?*

In the class of idempotent varieties satisfying $(\mathcal{V} \vee \mathcal{W})_{\mathcal{L}} \equiv \mathcal{V}_{\mathcal{L}} \vee \mathcal{W}_{\mathcal{L}}$, both being congruence modular and having a cube term are join prime.

Question

Does every $\vee$*-prime filter in the* $\mathcal{L}$*-interpretability lattice come from a coloring characterization?*

Question

Is there a coloring characterization for k-permutability for fixed k?

We know that $\mathcal{V}$ is not k-permutable for any k iff $\mathcal{V}$ is $(\{0,1\};\leq)$-colorable.

Let $\mathbb{P}_k = (\{1,\ldots,k\};\rightarrow)$ where $\alpha \rightarrow \beta$ iff $\alpha \leq \beta + 1$.

Then $\mathcal{V}$ is congruence $(k-1)$-permutable iff it is not $\mathbb{P}_k$-colorable.

Let $\hat{\mathbf{2}} = \left(\{0,1\}; (\{0,1\}^n \setminus \{1\}^n)_{n\in\omega}\right)$ and $\hat{\mathbb{2}} = \langle \{0,1\}; \not\to \rangle$.

Theorem

For idempotent $\mathcal{V}$ the following are equivalent.

- *$\mathcal{V}$ has no cube term,*
- *$\mathcal{V}$ is $\hat{\mathbf{2}}$-colorable,*
- *$\mathcal{V} \leq_{\mathcal{L}} \mathcal{V}(\hat{\mathbb{2}})$,*
- *$\mathcal{V}_{\mathcal{L}} \leq \mathcal{V}(\hat{\mathbb{2}})_{\mathcal{L}}$.*

If $\mathcal{V} = \mathcal{V}(\mathbb{A})$ for finite $\mathbb{A}$, then these are equivalent to

- *there exists $\mathbb{C} < \mathbb{D} \leq \mathbb{A}$ such that $D^n \setminus (D \setminus C)^n \leq \mathbb{A}^n$ for all n (Markovic, Maroti, McKenzie).*

Question

Without repeating the proof of MMM, can we obtain the finite algebra result from the other results?

Conclusion

Let $\hat{\mathbf{2}} = \left(\{0,1\}; (\{0,1\}^n \setminus \{1\}^n)_{n\in\omega}\right)$ and $\hat{\mathbb{2}} = \langle \{0,1\}; \not\to \rangle$.

Theorem

For idempotent $\mathcal{V}$ the following are equivalent.

- *$\mathcal{V}$ has no cube term,*
- *$\mathcal{V}$ is $\hat{\mathbf{2}}$-colorable,*
- *$\mathcal{V} \leq_{\mathcal{L}} \mathcal{V}(\hat{\mathbb{2}})$,*
- *$\mathcal{V}_{\mathcal{L}} \leq \mathcal{V}(\hat{\mathbb{2}})_{\mathcal{L}}$.*

Thank you.