Decidability in Universal Algebra

Matthew Moore

McMaster University

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Decidability in universal algebra

1 Introduction

2 Minsky machines

3 Results

4 $A(M)$

5 Conclusion
The following are known to be undecidable:

1. Whether \( A \) has a finite residual bound [McKenzie].
2. Whether \( A \) has a finite equational base [McKenzie; Willard].
3. Whether \( A \) has definable principal subcongruences [M].
4. Whether \( A \) has an NU term on \( A - \{p, q\} \) [Maroti].
5. Whether \( \text{typ}(\mathcal{V}(A)) \) contains \( i \) for \( i \in \{2, 3, 4, 5\} \) [Wood, McKenzie].
Conjectures

The following are conjectured to be undecidable for finite $\mathbb{A}$:

1. Whether $\mathbb{A}$ is finitely related.
2. Whether $\mathbb{A}$ is naturally dualizable.
3. + many more problems from clone theory.

If $\mathcal{V}(\mathbb{A})$...

- is congruence distributive, then we can decide 1 and 2.
- is congruence modular, then we can decide 1 and (sort of) 2.
- is congruence SD(meet), we have no strong results.
- has a compatible semilattice term, then we can decide both (‘yes’).
Decidability in universal algebra

1. Introduction

2. Minsky machines

3. Results

4. $\mathcal{A}(\mathcal{M})$

5. Conclusion
The Minsky machine is a simple model of computation that is equivalent to Turing machine computation.

A Minsky machine has

- states $0, 1, \ldots, N$ (state 1 is starting state, 0 is halting),
- registers $A$ and $B$ that have integer values $\geq 0$,
- instructions of the form $(i, R, j)$, meaning “in state $i$, increase register $R$ by 1 and enter $j$”.
- instructions of the form $(i, R, j, k)$, meaning “in state $i$, if $R$ is 0 enter $j$, otherwise decrease $R$ by 1 and enter $k$”.
A Minsky machine

Let $\mathcal{M}$ have instructions

$$(1, B, 3, 2), \quad (2, A, 1), \quad (3, A, 4), \quad (4, A, 0).$$

Start the machine with register contents $A = 4$, $B = 3$.

(do example at the board)

$\mathcal{M}$ computes $A + B + 2$ and stores the result in $A$.

Minsky machines are simpler than Turing machines:

- no tape,
- no machine head,
- instructions are more condensed.
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Definition

Let $A$ be an algebra, $a \in A$, and $R \leq A^n$. Define

$$R_a = \{ r \in R \mid \forall i \ r(i) \neq a \}.$$ 

This is said “$R$ sans $a$”.

For $S \leq A^n$, if

$$R_a = S_a.$$ 

then we write $R =_a S$ and say “$R$ and $S$ are equal sans $a$”.
Definition

Let $A$ be an algebra and $C \subseteq \text{Rel}(A)$ and $R \in \text{Rel}_n(A)$.

- $f : A^n \rightarrow A$ sans $a$ preserves $R$ if
  $$\forall (r_1, \ldots, r_n \in R) \exists (S = a R) f(r_1, \ldots, r_n) \in S.$$  

- $C \vdash^a R$ if any function which sans $a$ preserves $C$ also sans $a$ preserves $R$.

- $C \vdash^d R$ if for all $X \subseteq A^m$ and all $f : X \rightarrow A$, if $f$ preserves $C$ sans $a$ then $f$ preserves $R$ sans $a$.

Definition

- $A$ is finitely related sans $a$ if $\exists n$ such that $\text{Rel}_n(A) \vdash^a \text{Rel}(A)$.

- $A$ is naturally dualizable sans $a$ if $\exists n$ such that $\text{Rel}_n(A) \vdash^d \text{Rel}(A)$. 
Theorems

Theorem

The following are equivalent for a Minsky machine $M$.

1. $M$ halts.
2. $A(M)$ is finitely related sans $(\times, \times, \times)$.
3. $A(M)$ is naturally dualizable sans $(\times, \times, \times)$.

Theorem

Let $A$ be an algebra and $\Gamma, \varphi$ computable functions such that $\Gamma(n) \subseteq A^{kn}$, $\varphi(n) \in A^{kn}$. The problem

Input: $A, \Gamma, \varphi$

Output: The truth value of $\exists n [\varphi(n) \in Sg^{A^{kn}}(\Gamma(n))]$

is undecidable.

Remark: $A(M)$ is needlessly complicated for proving this last theorem, but we pick it up for free along the way.
The set $A(\mathcal{M})$

Let
- $\Sigma = \{ i \mid i \text{ a state of } \mathcal{M} \} \cup \{ \times \}$,
- $C = \{ 0, A, B, 1 \}$,
- $\kappa = \Sigma \times C$.

The underlying set of $A(\mathcal{M})$ is $A(\mathcal{M}) = \kappa^3$. 
Some operations of $A(\mathcal{M})$

Define helper functions $\text{comp}, \text{comp}', \text{pass} : \kappa \to \kappa$ by

\[
\text{comp}(\langle i, c \rangle) = \begin{cases} 
\langle j, R \rangle & \text{if } (i, R, j) \text{ is an instruction of } \mathcal{M} \text{ and } c = 0, \\
\langle k, 1 \rangle & \text{if } (i, R, j, k) \text{ is an instruction of } \mathcal{M} \text{ and } c = R, \\
\times & \text{otherwise,}
\end{cases}
\]

\[
\text{pass}(\langle i, c \rangle) = \begin{cases} 
\langle j, c \rangle & \text{if } (i, R, j) \text{ is an instruction of } \mathcal{M}, \\
\langle k, c \rangle & \text{if } (i, R, j, k) \text{ is an instruction of } \mathcal{M}, \\
\times & \text{otherwise,}
\end{cases}
\]

\[
\text{comp}'(\langle i, c \rangle) = \begin{cases} 
\langle j, 1 \rangle & \text{if } (i, R, j, k) \text{ is an instruction of } \mathcal{M} \text{ and } c = 1, \\
\times & \text{otherwise,}
\end{cases}
\]

\[
\text{pass}'(\langle i, c \rangle) = \begin{cases} 
\langle j, c \rangle & \text{if } (i, R, j, k) \text{ is an instruction of } \mathcal{M} \text{ and } c \neq R, \\
\times & \text{otherwise,}
\end{cases}
\]

$\text{comp}(\times) = \text{pass}(\times) = \text{comp}'(\times) = \text{pass}'(\times) = \times$. 

Some operations of $A(M)$

Let

$$M(x, y) = \begin{cases} (\times, \times, \times) & \text{if } y = (\times, \times, \times), \\ (\text{comp}(x_1), x_2, x_3) & \text{if } y_1 = x_3 \neq x_2 \text{ and } \text{comp}(x_1) \neq \times, \\ (\text{pass}(x_1), x_2, x_3) & \text{if } y_1 = x_2 \neq x_3, \\ (\times, \times, \times) & \text{otherwise}, \end{cases}$$

$$M'(x, y) = \begin{cases} (\times, \times, \times) & \text{if } y = (\times, \times, \times), \\ (\text{comp}'(x_1), x_2, x_3) & \text{if } y_1 = x_3 \neq x_2 \text{ and } \text{comp}'(x_1) \neq \times, \\ (\text{pass}'(x_1), x_2, x_3) & \text{if } y_1 = x_2 \neq x_3, \\ (\times, \times, \times) & \text{otherwise}. \end{cases}$$

*(do example at the board)*
Decidability in general algebra

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4 $\Lambda(\mathcal{M})$

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Theorems

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Thank you.