Decidability in Universal Algebra

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Known Results

The following are known to be undecidable:

1. Whether A has a finite residual bound [McKenzie].
2. Whether $\mathbb{A}$ has a finite equational base [McKenzie; Willard].
3. Whether $\mathbb{A}$ has definable principal subcongruences [M].
4. Whether $\mathbb{A}$ has an NU term on $A - \{p, q\}$ [Maroti].
5. Whether typ($\mathcal{V}(\mathbb{A})$) contains i for $i \in \{\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ [Wood, McKenzie]

Conjectures

The following are **conjectured** to be undecidable for finite $\mathbb{A}$:

1. Whether $\mathbb{A}$ is finitely related.
2. Whether $\mathbb{A}$ is naturally dualizable.
3. $+$ **many** more problems from clone theory.

If $\mathcal{V}(\mathbb{A})$...

- is congruence distributive, then we can decide 1 and 2.
- is congruence modular, then we can decide 1 and (sort of) 2.
- is congruence SD(meet), we have no strong results.
- has a compatible semilattice term, then we can decide both ('yes').

Minsky machines

The Minsky machine is a simple model of computation that is equivalent to Turing machine computation.

A Minsky machines has

- states $0, 1, \ldots, N$ (state 1 is starting state, 0 is halting),
- registers A and B that have integer values ≥ 0,
- instructions of the form (i, R, j), meaning
  "in state i, increase register R by 1 and enter j".
- instructions of the form (i, R, j, k), meaning
  "in state i, if R is 0 enter j, otherwise decrease R by 1 and enter k".

A Minsky machine

Let $\mathcal{M}$ have instructions

$$(1,B,3,2), \qquad (2,A,1), \qquad (3,A,4), \qquad (4,A,0).$$

Start the machine with register contents $A = 4$, $B = 3$.

(do example at the board)

$\mathcal{M}$ computes $A + B + 2$ and stores the result in A.

Minsky machines are simpler than Turing machines:

- no tape,
- no machine head,
- instructions are more condensed.

$\mathbb{R}$ sans a

Definition

Let $\mathbb{A}$ be an algebra, $a \in A$, and $\mathbb{R} \leq \mathbb{A}^n$. Define

$$R_a = \left\{ r \in R \mid \forall i \; r(i) \neq a \right\}.$$

This is said "$\mathbb{R}$ sans a".

For $\mathbb{S} \leq \mathbb{A}^n$, if

$$R_a = S_a.$$

then we write $\mathbb{R} =_a \mathbb{S}$ and say "$\mathbb{R}$ and $\mathbb{S}$ are equal sans a".

$\mathbb{R}$ sans a

Definition

Let $\mathbb{A}$ be an algebra and $\mathcal{C} \subseteq \mathrm{Rel}(\mathbb{A})$ and $\mathbb{R} \in \mathrm{Rel}_n(A)$.

- $f : A^n \to A$ sans a preserves $\mathbb{R}$ if
$$\forall(r_1, \dots, r_n \in R)\ \exists(\mathbb{S} =_a \mathbb{R})\ f(r_1, \dots, r_n) \in S.$$
- $\mathcal{C} \vdash^a \mathbb{R}$ if any function which sans a preserves $\mathcal{C}$ also sans a preserves $\mathbb{R}$.
- $\mathcal{C} \vdash^a_d \mathbb{R}$ if for all $X \subseteq A^m$ and all $f : X \to A$, if f preserves $\mathcal{C}$ sans a then f preserves $\mathbb{R}$ sans a.

Definition

- $\mathbb{A}$ is finitely related sans a if $\exists n$ such that $\mathrm{Rel}_n(\mathbb{A}) \vdash^a \mathrm{Rel}(\mathbb{A})$.
- $\mathbb{A}$ is naturally dualizable sans a if $\exists n$ such that $\mathrm{Rel}_n(\mathbb{A}) \vdash^a_d \mathrm{Rel}(\mathbb{A})$.

Theorems

Theorem

The following are equivalent for a Minksy machine $\mathcal{M}$.

1. *$\mathcal{M}$ halts.*
2. *$\mathbb{A}(\mathcal{M})$ is finitely related sans $(\times, \times, \times)$.*
3. *$\mathbb{A}(\mathcal{M})$ is naturally dualizable sans $(\times, \times, \times)$.*

Theorem

Let $\mathbb{A}$ be an algebra and Γ, φ computable functions such that $\Gamma(n) \subseteq A^{k_n}$, $\varphi(n) \in A^{k_n}$. The problem

$$\begin{array}{ll} \mathtt{Input:} & \mathbb{A},\ \Gamma,\ \varphi \\ \mathtt{Output:} & \text{The truth value of } \exists n\left[\varphi(n) \in \mathrm{Sg}^{\mathbb{A}^{k_n}}(\Gamma(n))\right] \end{array}$$

is undecidable.

Remark: $\mathbb{A}(\mathcal{M})$ is needlessly complicated for proving this last theorem, but we pick it up for free along the way.

The set $A(\mathcal{M})$

Let

- $\Sigma = \{i \mid \text{i a state of } \mathcal{M}\} \cup \{\times\}$,
- $C = \{0, A, B, 1\}$,
- $\kappa = \Sigma \times C$.

The underlying set of $\mathbb{A}(\mathcal{M})$ is $A(\mathcal{M}) = \kappa^3$.

Some operations of $A(\mathcal{M})$

Define helper functions $\text{comp}, \text{comp}', \text{pass} : \kappa \to \kappa$ by

$$\text{comp}(\langle i, c\rangle) = \begin{cases} \langle j, R\rangle & \text{if } (i, R, j) \text{ is an instruction of } \mathcal{M} \text{ and } c = 0, \\ \langle k, 1\rangle & \text{if } (i, R, j, k) \text{ is an instruction of } \mathcal{M} \text{ and } c = R, \\ \times & \text{otherwise,} \end{cases}$$

$$\text{pass}(\langle i, c\rangle) = \begin{cases} \langle j, c\rangle & \text{if } (i, R, j) \text{ is an instruction of } \mathcal{M}, \\ \langle k, c\rangle & \text{if } (i, R, j, k) \text{ is an instruction of } \mathcal{M}, \\ \times & \text{otherwise,} \end{cases}$$

$$\text{comp}'(\langle i, c\rangle) = \begin{cases} \langle j, 1\rangle & \text{if } (i, R, j, k) \text{ is an instruction of } \mathcal{M} \text{ and } c = 1, \\ \times & \text{otherwise,} \end{cases}$$

$$\text{pass}'(\langle i, c\rangle) = \begin{cases} \langle j, c\rangle & \text{if } (i, R, j, k) \text{ is an instruction of } \mathcal{M} \text{ and } c \neq R, \\ \times & \text{otherwise,} \end{cases}$$

$$\text{comp}(\times) = \text{pass}(\times) = \text{comp}'(\times) = \text{pass}'(\times) = \times.$$

Some operations of $A(\mathcal{M})$

Let

$$M(x,y) = \begin{cases} (\times,\times,\times) & \text{if } y = (\times,\times,\times), \\ (\mathrm{comp}(x_1), x_2, x_3) & \text{if } y_1 = x_3 \neq x_2 \text{ and } \mathrm{comp}(x_1) \neq \times, \\ (\mathrm{pass}(x_1), x_2, x_3) & \text{if } y_1 = x_2 \neq x_3, \\ (\times,\times,\times) & \text{otherwise,} \end{cases}$$

$$M'(x,y) = \begin{cases} (\times,\times,\times) & \text{if } y = (\times,\times,\times), \\ (\mathrm{comp}'(x_1), x_2, x_3) & \text{if } y_1 = x_3 \neq x_2 \text{ and } \mathrm{comp}'(x_1) \neq \times, \\ (\mathrm{pass}'(x_1), x_2, x_3) & \text{if } y_1 = x_2 \neq x_3, \\ (\times,\times,\times) & \text{otherwise.} \end{cases}$$

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is undecidable.

Thank you.