Naturally dualizable algebras omitting types **1** and **5** have a cube term

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Cube terms

An *n-dimensional cube term* for an algebra $\mathbb{A}$ is a term $t(\cdots)$ such that $\mathbb{A} \models t(u_1, \ldots, u_m) \approx \overline{x}$ for some $u_i \in \{x, y\}^n \setminus \{x\}^n$.

Example

$$\mathbb{A} \models t \begin{pmatrix} y & x & x & y & y & x & y \\ x & y & x & y & x & y & y \\ x & x & y & x & y & y & y \end{pmatrix} \approx \begin{pmatrix} x \\ x \\ x \end{pmatrix}$$

Finite idempotent $\mathbb{A}$ has a cube term...

$\Leftrightarrow$ $\mathbb{A}$ has few subpowers: $\exists k\ \forall n\ \log_2 |\mathbf{S}(\mathbb{A}^n)| \in \mathcal{O}(n^k)$.

$\Leftrightarrow$ $\mathbb{A}$ is congruence modular and finitely related.

$\Leftrightarrow$ $\mathbb{A}$ has no cube term blockers: $\mathbb{D} < \mathbb{C} \leq \mathbb{A}$ with $C^n \setminus (C \setminus D)^n \leq \mathbb{A}^n$.

$\Rightarrow$ CSP($\mathbb{A}$) is tractable.

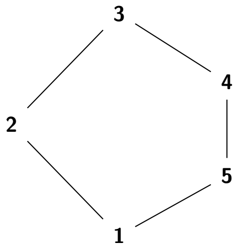
A little bit of TCT never hurt anybody

Let $\mathbb{A}$ be a finite algebra.

Let $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$ and let β cover α: $\begin{array}{c}\beta \\ | \\ \alpha\end{array}$

At a local level, it turns out that $\mathbb{A}$ 'behaves' in 1 of 5 ways on the cover:

1. G-set (unary): $\langle S; (\lambda_g)_{g \in G} \rangle$
2. Affine space: $\langle V; 0, +, -, (\lambda_f)_{f \in F} \rangle$
3. Boolean lattice: $\langle \{0, 1\}; \vee, \wedge, \neg \rangle$
4. Lattice: $\langle \{0, 1\}; \vee, \wedge \rangle$
5. Semilattice: $\langle \{0, 1\}; \vee \rangle$



This is the type of the cover (α, β).

A little bit of TCT never hurt anybody

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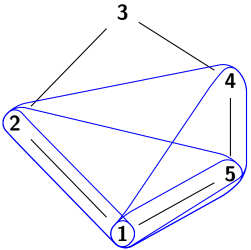
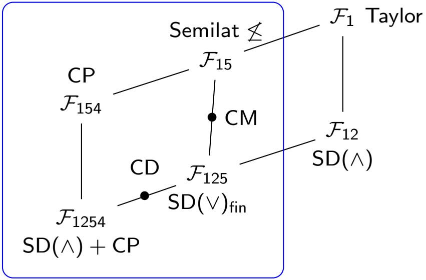


Many structural properties correspond to which types are forbidden in $\mathcal{V}_{\text{fin}}$.



Natural dualities

Motivational examples to think of:

- Stone duality between Boolean algebras and Stone spaces
- Priestley duality between distributive lattices and bounded Priestley spaces
- Pontryagin duality for abelian groups

Ingredients for a duality:

- a finite algebra $\mathbb{A} = \langle A; F \rangle$
- a structured topological space $\underset{\sim}{\mathbf{A}} = \langle A; G, H, R, \mathcal{T} \rangle$ ($G =$ total ops, $H =$ partial ops, $R =$ relations, $\mathcal{T} =$ discrete topology)
- If $\mathbb{B} \in \mathbf{SP}(\mathbb{A})$ then $\mathbb{B}^\partial = \mathrm{Hom}(\mathbb{B}, \mathbb{A}) \in \mathbf{S_cP_+}(\underset{\sim}{\mathbf{A}})$
- If $\underset{\sim}{\mathbf{B}} \in \mathbf{S_cP_+}(\underset{\sim}{\mathbf{A}})$ then $\underset{\sim}{\mathbf{B}}^\partial = \mathrm{Hom}(\underset{\sim}{\mathbf{B}}, \underset{\sim}{\mathbf{A}}) \in \mathbf{SP}(\mathbb{A})$
- $\underset{\sim}{\mathbf{A}}$ should be algebraic over $\mathbb{A}$

Natural dualities

For $\mathbb{B} \in \mathbf{SP}(\mathbb{A})$ there are mappings

$$\begin{array}{rrcl} e_{\mathbb{B}}: & \mathbb{B} & \to & \mathbb{B}^{\partial\partial} \\ & a & \mapsto & \left(e_{\mathbb{B}}(a) : \mathbb{B}^{\partial} \to \underset{\sim}{\mathbf{A}} : y \mapsto y(a)\right), \end{array}$$

$\underset{\sim}{\mathbf{A}}$ dualizes $\mathbb{A}$ if $e_{\mathbb{B}}$ are isomorphisms for all $\mathbb{B} \in \mathbf{SP}(\mathbb{A})$.

Examples of naturally dualizable algebras:

- Groups whose Sylow subgroups are abelian [Quackenbush, Szabo 2002], [Nickodemus 2007]
- Commutative rings whose Jacobson radical squares to (0) [Clark, Idziak, Sabourin, Szabo, Willard 2001]
- Algebras with near unanimity term [Davey, Heindorf, McKenzie 1995]
- Algebras with a compatible semilattice operation [Davey, Jackson, Pitkethly, Taukder 2007]
- Algebras with a cube term that satisfy the split centralizer condition [Kearnes, Szendrei]

Theorem

Let $\mathbb{A}$ be a finite algebra such that $\mathcal{V}(\mathbb{A})$ omits types $\mathbf{1}$ and $\mathbf{5}$. If $\mathbb{A}$ admits a natural duality, then $\mathbb{A}$ has a cube term.

Inherent non-dualizability

A finite algebra $\mathbb{A}$ is inherently non-dualizable if for all finite algebras $\mathbb{B}$,

$$\mathbb{A} \in \mathbf{SP}(\mathbb{B}) \qquad \Longrightarrow \qquad \mathbb{B} \text{ non-dualizable.}$$

Theorem (Davey, Idziak, Lampe, McNulty 1997)

Let $\mathbb{A}$ be a finite algebra, $\mathbb{B} \leq \mathbb{A}^Z$, $B_0 \subseteq B$ be an infinite subset such that

1. *there is a function $\varphi : \omega \to \omega$ such that for all $k \in \omega$ and all $\theta \in \mathrm{Con}(\mathbb{B})$ of index at most k, $\theta|_{B_0}$ has a unique block of size greater than $\varphi(k)$; and*
2. *if the element $g \in A^Z$ is defined by $g(z) = \pi(a_z)$ for $z \in Z$, where a_z is an element of the unique block of $\ker(\pi_z)|_{B_0}$ of size greater than $\varphi(|B|)$, then $g \notin B$.*

Then $\mathbb{A}$ is inherently non-dualizable.

Elements of the proof (assuming idempotency)

Pick $\mathbb{A}$ the smallest idempotent dualizable algebra omitting **1** and **5** without a cube term. $\mathbb{A}$ has a cube term blocker, so there are $a, b \in A$ such that there is no $t(\cdots)$ with

$$t\begin{pmatrix} \text{vectors in } a, b \\ \text{not all a} \\ \end{pmatrix} = \begin{pmatrix} a \\ \vdots \\ \vdots \\ a \end{pmatrix}.$$

Define elements of A^ω

$$\alpha_{i_1 \cdots i_n} = \begin{cases} b & \text{if } i \in \{i_1, \ldots, i_n\} \\ a & \text{otherwise.} \end{cases}$$

We will apply the non-dualizability theorem to

$$C_0 = \{\alpha_i \mid i \in \omega\} \qquad \mathbb{C} = \mathrm{Sg}^{\mathbb{A}^\omega}(C_0)$$

Elements of the proof (assuming idempotency)

To show:

1. there is a function $\varphi : \omega \to \omega$ such that for all $k \in \omega$ and all $\theta \in \mathrm{Con}(\mathbb{C})$ of index at most k, $\theta|_{C_0}$ has a unique block of size greater than $\varphi(k)$

Take $\varphi(k) = 1$, $\theta \in \mathrm{Con}(\mathbb{C})$, and assume that there are two $\theta|_{C_0}$-blocks of size > 1:

$$S = \{\alpha_1, \alpha_3, \dots\} \qquad \text{and} \qquad T = \{\alpha_2, \alpha_4, \dots\}$$

Claim

The set $\{\alpha_{12}, \alpha_{14}, \alpha_{23}, \alpha_{34}, \alpha_{124}, \alpha_{123}, \alpha_{134}, \alpha_{234}\}$ lies in single θ-block.

Proof of claim.

... (use the WNU and cube term blockers) ... ∎

Elements of the proof (assuming idempotency)

To show:

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Claim

$\alpha_1 \; \theta \; \alpha_2$.

Claim

$\alpha_1 \ \theta \ \alpha_2$.

Proof of claim.

$\mathbb{A}$ omits **1** and **5**, so we have terms

1. $\mathcal{V} \models f_0(x, y, u, v) \approx x$ and $\mathcal{V} \models f_{2m+1}(x, y, u, v) \approx v$,
2. $\mathcal{V} \models f_i(x, y, y, y) \approx f_{i+1}(x, y, y, y)$ for all even i,
3. $\mathcal{V} \models f_i(x, x, y, y) \approx f_{i+1}(x, x, y, y)$ for all odd i, and
4. $\mathcal{V} \models f_i(x, y, x, y) \approx f_{i+1}(x, y, x, y)$ for all odd i.

If i is even then $f_i(\alpha_1, \alpha_{12}, \alpha_{12}, \alpha_{12}) = f_{i+1}(\alpha_1, \alpha_{12}, \alpha_{12}, \alpha_{12})$.
If i is odd then by the previous claim,

$$f_i(\alpha_1, \alpha_{12}, \alpha_{12}, \alpha_{12}) \ \theta \ f_i(\alpha_1, \alpha_{12}, \alpha_{34}, \alpha_{234})$$

$$= f_i \begin{pmatrix} b & b & a & a \\ a & b & a & b \\ a & a & b & b \\ a & a & b & b \end{pmatrix} = f_{i+1} \begin{pmatrix} b & b & a & a \\ a & b & a & b \\ a & a & b & b \\ a & a & b & b \end{pmatrix}$$

$$= f_{i+1}(\alpha_1, \alpha_{12}, \alpha_{34}, \alpha_{234}) \ \theta \ f_{i+1}(\alpha_1, \alpha_{12}, \alpha_{12}, \alpha_{12}). \quad \bullet$$

Elements of the proof (assuming idempotency)

To show:

2. if the element $g \in A^\omega$ is defined by $g(z) = \pi_z(a_z)$ for $z \in \omega$, where a_z is an element of the unique block of $\ker(\pi_z)|_{C_0}$ of size greater than $\varphi(|C|)$, then $g \notin C$

Observe:

- $\ker(\pi_z)|_{C_0}$ has two blocks: $X_z = \{\alpha_z\}$ and $Y_z = \{\alpha_i \mid i \neq z\}$.
- Therefore Y_z is the unique large block, and $\pi_z(\alpha_i) = a$ for $i \neq z$.
- Thus $g(z) = a$ for all $z \in \omega$.

Elements of the proof (assuming idempotency)

Claim

$g \notin C$

Proof of claim.

Suppose that $g \in C$. Then there is a term $t(\cdots)$ such that $t(\alpha_1, \dots, \alpha_m) = g$. That is,

$$t \begin{pmatrix} b & a & \cdots & a \\ a & b & \cdots & a \\ \vdots & & \ddots & \vdots \\ a & a & \cdots & b \end{pmatrix} = \begin{pmatrix} a \\ a \\ \vdots \\ a \end{pmatrix}$$

We chose a and b so that this couldn't happen! ∎

By the non-dualizability theorem $\mathbb{A}$ is inherently non-dualizable. This is a contradiction.

Beyond finite algebras

We required finiteness in 2 places:

- tame congruence theory,
- a characterization having a cube term in terms of blockers.

Kearnes and Kiss extend some of the TCT results to the infinite setting. They replace

"$\mathcal{V}$ omits $\mathbf{1}$ and $\mathbf{5}$"

with

"$\mathcal{V}$ satisfies a nontrivial Maltsev condition which fails in some semilattice"

Recently the blockers results has been extended to infinite algebras by Kearnes, Szendrei and (independently) McKenzie, M.

A more general theorem

Theorem

Let $\mathbb{A}$ be an algebra that satisfies a nontrivial Maltsev condition which does not hold for some semilattice. If $\mathbb{A}$ admits a natural duality, then $\mathbb{A}$ has a cube term.

What is left?

$$(\text{cube term}) \implies \text{CM} \implies \text{omits } \mathbf{1}, \mathbf{5}$$

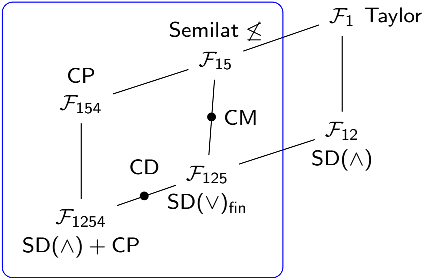
$$\begin{aligned}
(\text{cube term}) + (\text{split centralizer condition}) &\implies \text{dualizable} \\
(\text{cube term}) &\impliedby \text{dualizable, omits } \mathbf{1}, \mathbf{5}
\end{aligned}$$

The group $\mathbb{S}_3$ is in a CM variety, has a cube term, and is dualizable, but the algebra obtained by adding constants is non-dualizable.

Question

Does being dualizable and congruence modular imply the split centralizer condition, or is there something weaker?

What is left?



Question

What can be said about SD($\wedge$) algebras?

- Davey, Jackson, Pitkethly, Talukder prove that algebras with a compatible semilattice operation are dualizable.
- Davey, Idziak, Lampe, McNulty prove a graph algebra is dualizable iff every connected component is complete or bipartite complete.

What is left?

Question

Is it decidable given a finite $\mathbb{A}$ whether $\mathbb{A}$ is dualizable?

Probably 'yes' for algebras omitting $\mathbf{1}, \mathbf{5}$. A good place to look for this problem and the previous is in SD($\wedge$) algebras.

If we instead ask about partial algebras, then the answer is 'no'.

Theorem

Let $\mathbb{A}$ be a finite algebra such that $\mathcal{V}(\mathbb{A})$ omits types $\mathbf{1}$ and $\mathbf{5}$. If $\mathbb{A}$ admits a natural duality, then $\mathbb{A}$ has a cube term.

Theorem

Let $\mathbb{A}$ be an algebra that satisfies a nontrivial Maltsev identity which does not hold for some semilattice. If $\mathbb{A}$ admits a natural duality, then $\mathbb{A}$ has a cube term.

Thank you.