Indecision: finitely generated, finitely related clones
(finite degree clones are undecidable)

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1. Clones
2. The finite degree problem
3. The encoding of computation
4. Non-halting implies infinite degree
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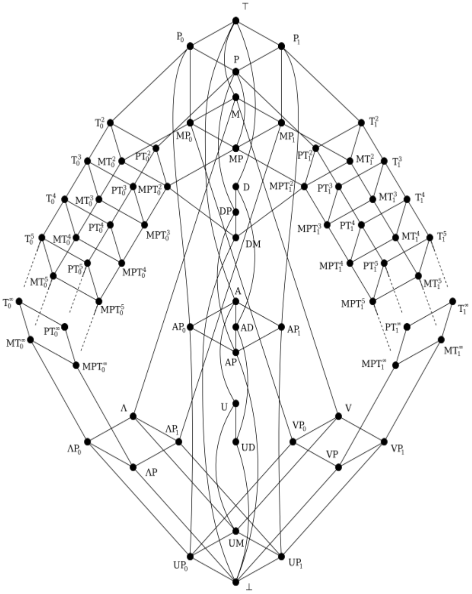
A **clone** is a set of finitary operations closed under

- composition,
- variable identification,
- variable permutation,
- introduction of extraneous variables.

Emil Post in 1941 famously classified all Boolean clones.

Over ($\geq$ 3)-element domains structure is quite complicated.



Clones are infinite. How can they be an input to an algorithm?

A clone on finite domain A can be **finitely specified** in essentially 2 ways.

First way: Given $\mathcal{F}$, a finite set of operations of A, define $\mathrm{Clo}(\mathcal{F})=$ "the smallest clone containing $\mathcal{F}$".

- A with $\mathcal{F}$ forms a algebra, $\mathbb{A}=\langle A;\mathcal{F}\rangle$. Define $\mathrm{Clo}(\mathbb{A})=\mathrm{Clo}(\mathcal{F})$.
- A **relation** of $\mathbb{A}$ is a subpower $R\subseteq A^n$ closed under $\mathcal{F}$ (hence $\mathrm{Clo}(\mathcal{F})$)
- Define $\mathrm{Rel}_n(\mathbb{A})=\mathrm{Rel}_n(\mathcal{F})=$ "all $(\leq n)$-ary relations of $\mathbb{A}$".
- Define $\mathrm{Rel}(\mathbb{A})=\mathrm{Rel}_n(\mathcal{F})=\bigcup_{n<\infty}\mathrm{Rel}_n(\mathbb{A})$

These are the **finitely generated** clones

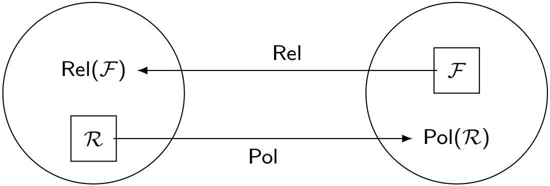
Second way: Given $\mathcal{R}$, a finite set of subpowers of A, define $\mathrm{Pol}(\mathcal{R})=$ "the set of all operations of A preserving all subpowers in $\mathcal{R}$".

These are the **finitely related**/**finite degree** clones.

$$\mathrm{Rel}(\mathcal{F}) = \left\{ R \subseteq A^n \mid R \text{ is preserved by all operations in } \mathcal{F} \right\}$$
$$\mathrm{Pol}(\mathcal{R}) = \left\{ f : A^n \to A \mid f \text{ preserves all subpowers in } \mathcal{R} \right\}$$

These two operators form a **Galois connection**.

$$\mathcal{R} \subseteq \mathrm{Rel}(\mathcal{F}) \iff \mathcal{F} \subseteq \mathrm{Pol}(\mathcal{R})$$



Every Galois connection defines two closure operators. Here, they are

$$\mathrm{Clo} = \mathrm{Pol} \circ \mathrm{Rel} \qquad \text{and} \qquad \mathrm{RClo} = \mathrm{Rel} \circ \mathrm{Pol}\,.$$

If $\mathbb{R} \in \mathrm{RClo}(\mathcal{S})$, then we say "$\mathcal{S}$ entails $\mathbb{R}$" and write $\mathcal{S} \models \mathbb{R}$

If $f \in \mathrm{Pol}(\mathcal{S})$, then we say "$\mathcal{S}$ entails f" and write $\mathcal{S} \models f$.

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For a set of relations $\mathcal{S}$, define

$$\deg(\mathcal{S}) = \sup\left\{\text{arity}(\mathbb{R}) \mid \mathbb{R} \in \mathcal{S}\right\}.$$

For a clone $\mathcal{C}$, define

$$\deg(\mathcal{C}) = \inf\left\{\deg(\mathcal{S}) \mid \text{Pol}(\mathcal{S}) = \mathcal{C}\right\}.$$

For an algebra $\mathbb{A}$, define

$$\deg(\mathbb{A}) = \deg(\text{Clo}(\mathbb{A})).$$

The Finite Degree Problem

`Input:` $\mathcal{F}$, a finite set of operations on a finite domain.

`Output:` whether $\deg(\text{Clo}(\mathcal{F})) < \infty$.

(Seems to originate in the 70s with the study of lattices of clones over more than 2 element domains.)

The Finite Degree Problem

`Input:` $\mathcal{F}$, a finite set of operations on a finite domain.
`Output:` whether $\deg(\mathrm{Clo}(F)) < \infty$.

Given a Minsky machine $\mathcal{M}$, we encode it into a finite algebra $\mathbb{A}(\mathcal{M})$.

Theorem

The following are equivalent.

- $\mathcal{M}$ *halts,*
- $\deg(\mathbb{A}(\mathcal{M})) < \infty$ *(i.e. $\mathbb{A}(\mathcal{M})$ is finitely related),*

Similar approaches have proved the following are undecidable:

- finite residual bound (McKenzie)
- finite axiomatizability/Tarski's problem (McKenzie)
- existence of a term op. that is NU on all but 2 elements (Maroti)
- DPSC, leading to another solution to Tarski's problem (M)
- profiniteness (Nurakunov and Stronkowski)

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A Minsky machine has...

- states $0, 1, \ldots, N$ (1 is the starting state, 0 is the halting state),
- registers A and B that have integer values ≥ 0,
- instructions of the form (i, R, j), meaning
  "in state i, increase register R by 1 and enter j".
- instructions of the form (i, R, j, k), meaning
  "in state i, if R is 0 enter j, otherwise decrease R by 1 and enter k".

In this talk, $\mathcal{M}$ is some fixed Minsky machine.

Let $\mathcal{M}$ have instructions

$$(1,B,3,2), \qquad (2,A,1), \qquad (3,A,4), \qquad (4,A,0).$$

Start the machine with register contents $A = 3$, $B = 2$.

Step	State	A	B
0	(1,	3,	2)
1	(2,	3,	1)
2	(1,	4,	1)
3	(2,	4,	0)

Step	State	A	B
4	(1,	5,	0)
5	(3,	5,	0)
6	(4,	6,	0)
7	(0,	7,	0)

$\mathcal{M}$ computes $A + B + 2$ and stores the result in A.

A **configuration** (i, α, β) represents each stage of computation.

Consider $\mathcal{M}$ as a function, and write

$$\mathcal{M}(i, \alpha, \beta) = (j, \alpha', \beta') \qquad \text{or} \qquad \mathcal{M}^n(i, \alpha, \beta) = (j, \alpha', \beta')$$

(single step of computation or multiple).

The **capacity** of $\mathcal{M}$ is the max sum of the registers.

The algebra $\mathbb{A}(\mathcal{M})$ has set

$$A(\mathcal{M}) = \{ \langle i, 0\rangle, \langle i, A\rangle, \langle i, B\rangle, \langle i, \bullet\rangle, \langle i, \times\rangle \mid i \text{ a state of } \mathcal{M} \}$$

The algebra is $\quad \mathbb{A}(\mathcal{M}) = \big\langle A(\mathcal{M})\ ;\ \wedge, M, M', I, H, N_0, S, N_\bullet, P \big\rangle.$

$$M(x, y) = \begin{cases} \langle j, R\rangle & \text{if } (i, R, j) \in \mathcal{M}, \\ & \qquad x = \langle i, \bullet\rangle, y = \langle i, 0\rangle; \\ \langle j, 0\rangle & \text{if } (i, R, k, j) \in \mathcal{M}, \\ & \qquad x = \langle i, \bullet\rangle, y = \langle i, R\rangle; \\ \langle j, c\rangle & \text{if } \big[(i, R, j) \text{ or } (i, R, k, j) \in \mathcal{M}\big], \\ & \qquad x = y = \langle i, c\rangle; \\ \langle j, c\rangle & \text{if } y = \langle i, c\rangle, M(y, x) = \langle j, d\rangle, \\ & \qquad d \neq \times, \text{ by above rules;} \\ \langle j, \times\rangle & \text{otherwise}^* \end{cases} \qquad M'(x) = \begin{cases} \langle k, c\rangle & \text{if } (i, R, k, j) \in \mathcal{M}, \\ & \qquad x = \langle i, c\rangle, c \neq R; \\ \langle k, \times\rangle & \text{otherwise}^* \end{cases}$$

- M encodes addition and subtraction operations.
- M' encodes testing for 0 in a register.
- $M(x, y) \neq \langle *, \times\rangle$ implies $x = y$ modulo a single coordinate transposition.

$$M(x,y)=\begin{cases}\langle j,R\rangle & \text{if } (i,R,j)\in\mathcal{M},\\ & \quad x=\langle i,\bullet\rangle,\ y=\langle i,0\rangle;\\ \langle j,0\rangle & \text{if } (i,R,k,j)\in\mathcal{M},\\ & \quad x=\langle i,\bullet\rangle,\ y=\langle i,R\rangle;\\ \langle j,c\rangle & \text{if } \big[(i,R,j) \text{ or } (i,R,k,j)\in\mathcal{M}\big],\\ & \quad x=y=\langle i,c\rangle;\\ \langle j,c\rangle & \text{if } y=\langle i,c\rangle,\ M(y,x)=\langle j,d\rangle,\\ & \quad d\neq\times,\ \text{by above rules};\\ \cdots & \cdots \quad \cdots \quad \cdots \quad \cdots\end{cases} \quad M'(x)=\begin{cases}\langle k,c\rangle & \text{if } (i,R,k,j)\in\mathcal{M},\\ & \quad x=\langle i,c\rangle,\ c\neq R;\\ \cdots & \cdots \quad \cdots \quad \cdots\end{cases}$$

Let $\mathcal{M}=\big\{(1,B,3,2),(2,A,1),(3,A,4),(4,A,0)\big\}$. ($A+B+2$ from before)

1: $M\begin{pmatrix}\langle 1,\bullet\rangle,\langle 1,B\rangle\\ \langle 1,0\rangle,\langle 1,0\rangle\\ \langle 1,0\rangle,\langle 1,0\rangle\\ \langle 1,B\rangle,\langle 1,\bullet\rangle\end{pmatrix}=\begin{pmatrix}\langle 2,0\rangle\\ \langle 2,0\rangle\\ \langle 2,0\rangle\\ \langle 2,\bullet\rangle\end{pmatrix}$ **4:** $M\begin{pmatrix}\langle 3,\bullet\rangle,\langle 3,0\rangle\\ \langle 3,0\rangle,\langle 3,\bullet\rangle\\ \langle 3,0\rangle,\langle 3,0\rangle\\ \langle 3,A\rangle,\langle 3,A\rangle\end{pmatrix}=\begin{pmatrix}\langle 4,A\rangle\\ \langle 4,\bullet\rangle\\ \langle 4,0\rangle\\ \langle 4,A\rangle\end{pmatrix}$

2: $M\begin{pmatrix}\langle 2,0\rangle,\langle 2,\bullet\rangle\\ \langle 2,0\rangle,\langle 2,0\rangle\\ \langle 2,0\rangle,\langle 2,0\rangle\\ \langle 2,\bullet\rangle,\langle 2,0\rangle\end{pmatrix}=\begin{pmatrix}\langle 1,\bullet\rangle\\ \langle 1,0\rangle\\ \langle 1,0\rangle\\ \langle 1,A\rangle\end{pmatrix}$ **5:** $M\begin{pmatrix}\langle 4,A\rangle,\langle 4,A\rangle\\ \langle 4,\bullet\rangle,\langle 4,0\rangle\\ \langle 4,0\rangle,\langle 4,\bullet\rangle\\ \langle 4,A\rangle,\langle 4,A\rangle\end{pmatrix}=\begin{pmatrix}\langle 0,A\rangle\\ \langle 0,A\rangle\\ \langle 0,\bullet\rangle\\ \langle 0,A\rangle\end{pmatrix}$

3: $M'\begin{pmatrix}\langle 1,\bullet\rangle\\ \langle 1,0\rangle\\ \langle 1,0\rangle\\ \langle 1,A\rangle\end{pmatrix}=\begin{pmatrix}\langle 3,\bullet\rangle\\ \langle 3,0\rangle\\ \langle 3,0\rangle\\ \langle 3,A\rangle\end{pmatrix}$

Step	State	A	B
0	1	0	1
1	2	0	0
2	1	1	0
3	3	1	0
4	4	2	0
5	0	3	0

Takeaways: on a relation $\mathbb{R} \leq \mathbb{A}(\mathcal{M})^n$ ...

- certain elements of R encode configurations of $\mathcal{M}$,
- M and M' encode the action of $\mathcal{M}$ in the presence of certain elements of R.

Questions

- What if R doesn't contain these kinds of elements?
- What if R contains elements that aren't "computational": multiple $\bullet$'s or non-constant states.

Given configuration (k, α, β), in $\mathbb{A}(\mathcal{M})^n$ define

$$\mathsf{c}(k,\alpha,\beta) = \bigcup_{p \in P_n} \left\{ p\Big(\langle k, \bullet \rangle, \underbrace{\langle k, A \rangle, \ldots, \langle k, A \rangle}_{\alpha}, \underbrace{\langle k, B \rangle, \ldots, \langle k, B \rangle}_{\beta}, \underbrace{\langle k, 0 \rangle, \ldots, \langle k, 0 \rangle}_{n-\alpha-\beta-1} \Big) \right\}$$

Call $\mathbb{R}$ **computational** if it doesn't contain any elements with 2 $\bullet$'s or non-constant state.

The **capacity** of computational $\mathbb{R}$ is (number of coordinates with $\bullet$)-1.

Let $\mathbb{S}_m = \mathrm{Sg}_{\mathbb{A}(\mathcal{M})^m}\big(\,\mathsf{c}(1,0,0)\big)$.

Theorem (The Coding Theorem)

- *If* $\mathcal{M}^n(1,0,0) = (k,\alpha,\beta)$ *and this computation has capacity* $m-1$*, then* $\mathsf{c}(k,\alpha,\beta) \subseteq S_m$.
- *If* $\mathsf{c}(k,\alpha,\beta) \subseteq S_m$ *and* $\mathcal{M}$ *does not halt with capacity* $m-1$ *then for some* n *we have* $\mathcal{M}^n(1,0,0) = (k,\alpha,\beta)$ *and this computation has capacity* $m-1$.

Corollary

The following are equivalent.

- $\mathcal{M}$ *halts with capacity* $m-1$,
- $\mathbb{S}_m$ *is halting,*
- *every computational* $\mathbb{R} \leq \mathbb{A}(\mathcal{M})^\ell$ *with capacity* $m-1$ *is halting.*

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Observe

$$\begin{aligned} \deg(\mathcal{C}) = \infty \quad & \text{if and only if} \quad \mathsf{Rel}_n(\mathcal{C}) \not\models \mathsf{Rel}(\mathcal{C}) \text{ for all } n \\ & \text{if and only if} \quad \mathsf{Rel}_n(\mathcal{C}) \not\models \mathbb{R} \text{ for all } n \text{ and some } \mathbb{R} \end{aligned}$$

Idea: to show that $\deg(\mathbb{A}(\mathcal{M})) = \infty$ when $\mathcal{M}$ does not halt, we prove the last equivalence for $\mathcal{C} = \mathsf{Clo}(\mathbb{A}(\mathcal{M}))$.

$\mathrm{Rel}_n(\mathcal{C}) \models \mathbb{R}$ if and only if $\mathbb{R}$ can be built from $\mathrm{Rel}_n(\mathcal{C})$, in finitely many steps, by applying the following constructions:

- intersection of equal arity relations,
- (cartesian) product of finitely many relations,
- permutation of the coordinates of a relation, and
- projection of a relation onto a subset of coordinates.

Theorem (Zadori 1995)

$\mathrm{Rel}_n(\mathbb{A}) \models \mathbb{S}$ *if and only if*

$$\mathbb{S} = \pi\left(\bigcap_{i\in I} \mu_i\Big(\prod_{j\in J_i} \mathbb{R}_{ij}\Big)\right)$$

for some $\mathbb{R}_{ij} \in \mathrm{Rel}_n(\mathbb{A})$*, some coordinate projection* π*, and some coordinate permutations* μ_i*.*

Lemma

Suppose that

$$\mathsf{c}(1,0,0) \subseteq \pi\left(\bigcap_{i\in I} \mu_i\Big(\prod_{j\in J_i} \mathbb{R}_{ij}\Big)\right) = \mathbb{S} \leq \mathbb{A}(\mathcal{M})^m,$$

where π is a projection, the μ_i are permutations, and the $\mathbb{R}_{ij}$ are a finite collection of members of $\mathrm{Rel}_n(\mathbb{A}(\mathcal{M}))$, and $n < m$. Then $\mathbb{S}$ is halting.

Theorem

The following hold for any Minsky machine $\mathcal{M}$.

- *If $\mathcal{M}$ does not halt with capacity m then $m < \deg(\mathbb{A}(\mathcal{M}))$.*
- *If $\mathcal{M}$ does not halt then $\mathbb{A}(\mathcal{M})$ is not finitely related.*

Proof: Suppose that $\deg(\mathbb{A}(\mathcal{M})) \leq m$. This implies in particular that $\mathrm{Rel}_m(\mathbb{A}(\mathcal{M})) \models \mathbb{S}_{m+1}$. By Zadori's theorem, $\mathbb{S}_{m+1}$ can be represented as in the Lemma above, so by that same Lemma it is halting. By the Coding Theorem, this implies that $\mathcal{M}$ halts with capacity m, a contradiction. ·

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Strategy

- The relations $\mathbb{S}_m$ witnessed non-entailment when $\mathcal{M}$ did not halt. When $\mathcal{M}$ does halt, these relations eventually witness the halting.
- Show that for some suitably chosen k, we have $\mathrm{Rel}_k(\mathbb{A}(\mathcal{M})) \models \mathrm{Rel}_n(\mathbb{A}(\mathcal{M}))$ for all n.
- We proceed with induction on n.
- The base case of $n = k$ is trivial.
- We thus endeavor to prove $\mathrm{Rel}_{n-1}(\mathbb{A}(\mathcal{M})) \models \mathbb{R}$ for $\mathbb{R} \in \mathrm{Rel}_n(\mathbb{A}(\mathcal{M}))$.
- Relations in $\mathrm{Rel}_n(\mathbb{A}(\mathcal{M}))$ can be divided into 4 different kinds, so we proceed by cases.
- Recall $\mathbb{A}(\mathcal{M}) = \langle A(\mathcal{M}) \,;\, \wedge, M, M', I, H, N_0, S, N_\bullet, P \rangle$. Different collections of operations handle entailment in each of the different cases.

$$\mathbb{A}(\mathcal{M}) = \langle A(\mathcal{M})\,;\ \wedge, M, M', I, H, N_0, S, N_\bullet, P \rangle$$

Case $\mathbb{R}$ is non-computational

- There is an element with 2 $\bullet$'s or with non-constant state.
- 2 $\bullet$'s: operation $N_\bullet$ handles entailment.
- Non-constant state: operation P handles entailment.

Theorem

If $m \geq 3$ and $\mathbb{R} \leq \mathbb{A}(\mathcal{M})^m$ is non-computational then $\mathrm{Rel}_{m-1}(\mathbb{A}(\mathcal{M})) \models \mathbb{R}$.

Case $\mathbb{R}$ is halting

- R contains $\mathsf{c}(0,0,0)$.
- Any element of $\mathsf{c}(0,0,0)$ can be used with operations I, H, and N_0 to prove entailment.

Theorem

If $m \geq 3$ and $\mathbb{R} \leq \mathbb{A}(\mathcal{M})^m$ is halting then $\mathrm{Rel}_{m-1}(\mathbb{A}(\mathcal{M})) \models \mathbb{R}$.

We are left to examine computational non-halting $\mathbb{R} \leq \mathbb{A}(\mathcal{M})^n$.

Let's say that $\mathcal{M}$ halts with capacity κ.

Two metrics: (both subsets of $[n]$)

- $\mathcal{D}(\mathbb{R}) =$ "coordinates i such that $r \in R$ with $r(i) = \langle j, \bullet \rangle$"
  $=$ "the $\bullet$ (dot) part of $\mathbb{R}$.
- $\mathcal{N}(\mathbb{R}) =$ "the inherently non-halting part of $\mathbb{R}$" ...
  - $\pi_{\mathcal{N}(\mathbb{R})}(\mathbb{R})$ is non-halting,
  - If $K = \left|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})\right|$ then $\mathbb{S}_K \leq \mathbb{R}$.

Case $\mathbb{R}$ is computational and $|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})| > \kappa$

- $\left|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})\right| > \kappa$ then $\mathbb{R}$ contains a halting subalgebra.
- it follows that $\mathbb{R}$ halts!

We thus consider computational non-halting $\mathbb{R}$ with $\left|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})\right| \leq \kappa$. ·

Case computational non-halting $\mathbb{R}$ with $\left|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})\right| \leq \kappa$

Theorem

Assume that $n \geq \kappa + 16$ and

- $\mathbb{R} \leq \mathbb{A}(\mathcal{M})^n$ *is computational non-halting,*
- $\left|\mathcal{N}(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})\right| \leq \kappa$,
- $\vdots$ *(several technical hypotheses)*

Then $\mathrm{Rel}_n(\mathbb{A}(\mathcal{M})) \models \mathbb{R}$.

This completes the case analysis!

Theorem

If $\mathcal{M}$ halts with capacity κ then $\deg(\mathbb{A}(\mathcal{M})) \leq \kappa + 16$.

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Theorem

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- $\mathcal{M}$ *halts,*
- $\deg(\mathbb{A}(\mathcal{M})) < \infty$ *(i.e. $\mathbb{A}(\mathcal{M})$ is finitely related),*
- $\mathcal{M}$ *halts with capacity* $\deg(\mathbb{A}(\mathcal{M}))$.

Interesting observations

- There are infinitely many $\mathcal{M}$ and $n \in \mathbb{N}$ such that $\mathcal{M}$ halts in $\leq n$ steps but this is not provable in ZFC.
- Thus, there are infinitely many $\mathcal{M}$ and $n \in \mathbb{N}$ such that $\deg(\mathbb{A}(\mathcal{M})) \leq n$ is true but not provable in ZFC.
- There are finite algebras $\mathbb{A}$ that are finitely related but for which a bound on $\deg(\mathbb{A})$ cannot be proven.
- $\operatorname{maxdeg}_\sigma(n) = \sup\left\{ \deg(\mathbb{A}) \;\middle|\; \begin{array}{c} \mathbb{A} \text{ has signature } \sigma, \\ \deg(\mathbb{A}) < \infty, \text{ and } |A| \leq n \end{array} \right\}$

  not computable.

Problem (Finite Generation Problem)

Given relations $\mathcal{R}$, decide if $\mathcal{C} = \mathrm{Pol}(\mathcal{R})$ is finitely generated. That is, whether $\mathcal{C} = \mathrm{Clo}(\mathcal{F})$ for some finite set of operations $\mathcal{F}$.

The theory of Natural Dualities for algebras has a very similar notion of relational entailment.

We can modify the definition of $\deg(\cdot)$ to obtain a duality degree: $\deg_\partial(\cdot)$.

Problem (Finite Duality Degree)

Decide whether $\deg_\partial(\mathbb{A}) < \infty$ for finite $\mathbb{A}$.

Duality entailment implies usual entailment, so we already have that $\mathbb{A}(\mathcal{M})$ is not finitely duality related when $\mathcal{M}$ does not halt.

Problem

If $\mathcal{M}$ halts, is $\deg_\partial(\mathbb{A}(\mathcal{M})) < \infty$?

Theorem

The following are equivalent.

- $\mathcal{M}$ *halts,*
- $\deg(\mathbb{A}(\mathcal{M})) < \infty$ *(i.e.* $\mathbb{A}(\mathcal{M})$ *is finitely related),*
- $\mathcal{M}$ *halts with capacity* $\deg(\mathbb{A}(\mathcal{M}))$.

"Finite Degree Clones are Undecidable"

Thank you for your attention.