Supernilpotence Need Not Imply Nilpotence

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Let $\mathbb{A}$ be an algebra and $\mathbb{R} \leq \mathbb{A}^4$.

$$\cancel{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}} \in R \qquad \text{instead, we will write} \qquad \begin{tikzcd} a \arrow[r, dash] \arrow[d, dash] & b \arrow[d, dash] \\ c \arrow[r, dash] & d \end{tikzcd} \in R$$

Let α_1, α_2 be congruences of $\mathbb{A}$. The (α_1, α_2)-**squares** of $\mathbb{A}$ are

$$\mathrm{M}(\alpha_1, \alpha_2) := \mathrm{Sg}_{\mathbb{A}^4} \left\{ \begin{tikzcd} a \arrow[r, dash] \arrow[d, dash] & b \arrow[d, dash] \\ a \arrow[r, dash] & b \end{tikzcd}, \begin{tikzcd} c \arrow[r, dash] \arrow[d, dash] & c \arrow[d, dash] \\ d \arrow[r, dash] & d \end{tikzcd} \;\middle|\; (a, b) \in \alpha_1,\ (c, d) \in \alpha_2 \right\}$$

Observe

$$\begin{tikzcd} a \arrow[r, dash] \arrow[d, dash] & b \arrow[d, dash] \\ c \arrow[r, dash] & d \end{tikzcd} \in \mathrm{M}(\alpha_1, \alpha_2) \qquad \Longrightarrow \qquad \begin{tikzcd} a \arrow[r, dash] \arrow[d, dash] \arrow[r, dash, bend left=60, "\alpha_1"] \arrow[d, dash, bend right=60, "\alpha_2"'] & b \arrow[d, dash] \arrow[d, dash, bend left=60, "\alpha_2"] \\ c \arrow[r, dash] \arrow[r, dash, bend right=60, "\alpha_1"'] & d \end{tikzcd}$$

$$\mathsf{M}(\alpha_1,\alpha_2) \coloneqq \mathsf{Sg}_{\mathbb{A}^4}\left\{ \begin{tikzcd}[arrows=dash] a \arrow[r] \arrow[d] & b \arrow[d] \\ a \arrow[r] & b \end{tikzcd}, \begin{tikzcd}[arrows=dash] c \arrow[r] \arrow[d] & c \arrow[d] \\ d \arrow[r] & d \end{tikzcd} \;\middle|\; (a,b)\in\alpha_1,\ (c,d)\in\alpha_2 \right\}$$

α_1 **centralizes** α_2 modulo δ if ... [write $\mathsf{C}(\alpha_1,\alpha_2;\delta)$]

$$\forall \begin{tikzcd}[arrows=dash] a \arrow[r] \arrow[d] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd} \in \mathsf{M}(\alpha_1,\alpha_2), \quad \begin{tikzcd}[arrows=dash] a \arrow[r] \arrow[d] \arrow[d, bend right=60, "\delta"'] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd} \quad\Longrightarrow\quad \begin{tikzcd}[arrows=dash] a \arrow[r] \arrow[d] & b \arrow[d] \arrow[d, dashed, bend left=60, "\delta"] \\ c \arrow[r] & d \end{tikzcd}$$

The **commutator** of α_1 and α_2 is the smallest δ satisfying the above,

$$[\alpha_1,\alpha_2] \coloneqq \bigwedge_{\mathsf{C}(\alpha_1,\alpha_2;\delta)} \delta$$

Can we generalize this to a higher arity commutator?

Let $\mathbb{A}$ be an algebra and $\mathbb{R} \leq \mathbb{A}^{2^3}$.

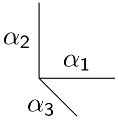
$$\cancel{\begin{pmatrix} a \\ \vdots \\ h \end{pmatrix} \in R} \qquad \text{instead, we will write} \qquad \begin{tikzcd}[row sep=small, column sep=small] a \arrow[rr, dash] \arrow[dd, dash] \arrow[dr, dash] & & b \arrow[dd, dotted, dash] \arrow[dr, dash] & \\ & e \arrow[rr, dash] \arrow[dd, dash] & & f \arrow[dd, dash] \\ c \arrow[rr, dotted, dash] \arrow[dr, dash] & & d \arrow[dr, dotted, dash] & \\ & g \arrow[rr, dash] & & h \end{tikzcd} \in R$$

Let $\alpha_1, \alpha_2, \alpha_3$ be congruences of $\mathbb{A}$. The $(\alpha_1, \alpha_2, \alpha_3)$-**cubes** of $\mathbb{A}$ are

$$\mathrm{M}(\alpha_1, \alpha_2, \alpha_3) := \mathrm{Sg}_{\mathbb{A}^{2^3}} \left\{ \begin{tikzcd}[row sep=small, column sep=small] a \arrow[rr, dash] \arrow[dd, dash] \arrow[dr, dash] & & b \arrow[dd, dotted, dash] \arrow[dr, dash] & \\ & a \arrow[rr, dash] \arrow[dd, dash] & & b \arrow[dd, dash] \\ a \arrow[rr, dotted, dash] \arrow[dr, dash] & & b \arrow[dr, dotted, dash] & \\ & a \arrow[rr, dash] & & b \end{tikzcd}, \begin{tikzcd}[row sep=small, column sep=small] c \arrow[rr, dash] \arrow[dd, dash] \arrow[dr, dash] & & c \arrow[dd, dotted, dash] \arrow[dr, dash] & \\ & c \arrow[rr, dash] \arrow[dd, dash] & & c \arrow[dd, dash] \\ d \arrow[rr, dotted, dash] \arrow[dr, dash] & & d \arrow[dr, dotted, dash] & \\ & d \arrow[rr, dash] & & d \end{tikzcd}, \begin{tikzcd}[row sep=small, column sep=small] e \arrow[rr, dash] \arrow[dd, dash] \arrow[dr, dash] & & e \arrow[dd, dotted, dash] \arrow[dr, dash] & \\ & f \arrow[rr, dash] \arrow[dd, dash] & & f \arrow[dd, dash] \\ e \arrow[rr, dotted, dash] \arrow[dr, dash] & & e \arrow[dr, dotted, dash] & \\ & f \arrow[rr, dash] & & f \end{tikzcd} \;\middle|\; \begin{array}{c} (a,b) \in \alpha_1, \\ (c,d) \in \alpha_2, \\ (e,f) \in \alpha_3 \end{array} \right\}$$

Observe

$$\begin{tikzcd}[row sep=small, column sep=small] a \arrow[rr, dash] \arrow[dd, dash] \arrow[dr, dash] & & b \arrow[dd, dotted, dash] \arrow[dr, dash] & \\ & e \arrow[rr, dash] \arrow[dd, dash] & & f \arrow[dd, dash] \\ c \arrow[rr, dotted, dash] \arrow[dr, dash] & & d \arrow[dr, dotted, dash] & \\ & g \arrow[rr, dash] & & h \end{tikzcd} \in \mathrm{M}(\alpha_1, \alpha_2, \alpha_3) \qquad \Longrightarrow$$



in

$$\begin{tikzcd}[row sep=small, column sep=small] a \arrow[rr, dash] \arrow[dd, dash] \arrow[dr, dash] & & b \arrow[dd, dotted, dash] \arrow[dr, dash] & \\ & e \arrow[rr, dash] \arrow[dd, dash] & & f \arrow[dd, dash] \\ c \arrow[rr, dotted, dash] \arrow[dr, dash] & & d \arrow[dr, dotted, dash] & \\ & g \arrow[rr, dash] & & h \end{tikzcd}$$

$$\mathsf{M}(\alpha_1,\alpha_2,\alpha_3) \coloneqq \mathrm{Sg}_{\mathbb{A}^{2^3}} \left\{ \begin{tikzcd}[row sep=small, column sep=small] a \arrow[rr, no head] \arrow[dr, no head] \arrow[dd, no head] & & b \arrow[dr, no head] \arrow[dd, no head, dotted] & \\ & a \arrow[rr, no head] \arrow[dd, no head] & & b \arrow[dd, no head] \\ a \arrow[rr, no head, dotted] \arrow[dr, no head] & & b \arrow[dr, no head, dotted] & \\ & a \arrow[rr, no head] & & b \end{tikzcd}, \begin{tikzcd}[row sep=small, column sep=small] c \arrow[rr, no head] \arrow[dr, no head] \arrow[dd, no head] & & c \arrow[dr, no head] \arrow[dd, no head, dotted] & \\ & c \arrow[rr, no head] \arrow[dd, no head] & & c \arrow[dd, no head] \\ d \arrow[rr, no head, dotted] \arrow[dr, no head] & & d \arrow[dr, no head, dotted] & \\ & d \arrow[rr, no head] & & d \end{tikzcd}, \begin{tikzcd}[row sep=small, column sep=small] e \arrow[rr, no head] \arrow[dr, no head] \arrow[dd, no head] & & e \arrow[dr, no head] \arrow[dd, no head, dotted] & \\ & f \arrow[rr, no head] \arrow[dd, no head] & & f \arrow[dd, no head] \\ e \arrow[rr, no head, dotted] \arrow[dr, no head] & & e \arrow[dr, no head, dotted] & \\ & f \arrow[rr, no head] & & f \end{tikzcd} \;\middle|\; \begin{array}{c} (a,b) \in \alpha_1, \\ (c,d) \in \alpha_2, \\ (e,f) \in \alpha_3 \end{array} \right\}$$

α_1, α_2 **centralize** α_3 modulo δ if . . . [write $\mathsf{C}(\alpha_1,\alpha_2,\alpha_3;\delta)$]

$$\forall \begin{tikzcd}[row sep=small, column sep=small] a \arrow[rr, no head] \arrow[dr, no head] \arrow[dd, no head] & & b \arrow[dr, no head] \arrow[dd, no head, dotted] & \\ & e \arrow[rr, no head] \arrow[dd, no head] & & f \arrow[dd, no head] \\ c \arrow[rr, no head, dotted] \arrow[dr, no head] & & d \arrow[dr, no head, dotted] & \\ & g \arrow[rr, no head] & & h \end{tikzcd} \in \mathsf{M}(\alpha_1,\alpha_2,\alpha_3), \qquad \begin{tikzcd}[row sep=small, column sep=small] a \arrow[rr, no head] \arrow[dr, no head] \arrow[dd, no head] \arrow[dr, no head, bend right=60, "\delta"'] & & b \arrow[dr, no head] \arrow[dd, no head, dotted] & \\ & e \arrow[rr, no head] \arrow[dd, no head] & & f \arrow[dd, no head] \\ c \arrow[rr, no head, dotted] \arrow[dr, no head] \arrow[dr, no head, bend right=60, "\delta"'] & & d \arrow[dr, no head, dotted] \arrow[dr, no head, bend left=60, "\delta"] & \\ & g \arrow[rr, no head] & & h \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd}[row sep=small, column sep=small] a \arrow[rr, no head] \arrow[dr, no head] \arrow[dd, no head] & & b \arrow[dr, no head] \arrow[dd, no head, dotted] \arrow[dr, no head, dashed, bend left=60, "\delta"] & \\ & e \arrow[rr, no head] \arrow[dd, no head] & & f \arrow[dd, no head] \\ c \arrow[rr, no head, dotted] \arrow[dr, no head] & & d \arrow[dr, no head, dotted] & \\ & g \arrow[rr, no head] & & h \end{tikzcd}$$

The **3-dim commutator** of $\alpha_1, \alpha_2, \alpha_3$ is the smallest δ satisfying the above,

$$[\alpha_1,\alpha_2,\alpha_3] \coloneqq \bigwedge_{\mathsf{C}(\alpha_1,\alpha_2,\alpha_3;\delta)} \delta$$

Can we generalize this to a higher arity commutator?

Yes, even for arbitrary n!

A general definition of the commutator yields general notions of abelianness, solvability, and nilpotence.

- $\mathbb{A}$ is **abelian** if $[\mathbb{1}, \mathbb{1}] = \mathbb{0}$.
- Define the **derived series**,

$$[\alpha]_0 \coloneqq \alpha \qquad\qquad [\alpha]_{n+1} \coloneqq \big[[\alpha]_n, [\alpha]_n\big].$$

  $\mathbb{A}$ is m-**solvable** if $[\mathbb{1}]_m = \mathbb{0}$ for some m.
- define the **lower central series**,

$$(\alpha]_0 \coloneqq \alpha \qquad\qquad (\alpha]_{n+1} \coloneqq \big[\alpha, (\alpha]_n\big].$$

  $\mathbb{A}$ is (left) m-**nilpotent** if $(\mathbb{1}]_m = \mathbb{0}$ for some m.
- The higher dimensional commutator has its own series,

$$\alpha \geq [\alpha, \alpha] \geq [\alpha, \alpha, \alpha] \geq \cdots \geq [\alpha, \ldots, \alpha] \geq \ldots$$

  $\mathbb{A}$ is $(m+1)$-**supernilpotent** if $[\mathbb{1}, \ldots, \mathbb{1}] = \mathbb{0}$ (m-ary)

($\mathbb{1}$ is the universal congruence and $\mathbb{0}$ is the trivial congruence)

For an algebra $\mathbb{A}$ we have general notions of

- abelianness
- solvability
- nilpotence
- supernilpotence

How are these related?

How are nilpotence and supernilpotence related?

Does supernilpotence imply nilpotence?

Theorem (Moorhead)

*If $\mathbb{A}$ satisfies a non-trivial idempotent equational condition, then "**Yes**".*

Theorem (Kearnes, Szendrei)

*If $\mathbb{A}$ is finite, then "**Yes**".*

Theorem (M+M)

Supernilpotence does not imply nilpotence in general.

Idea: Construct $\mathbb{A}$ as simply as possible.

- Make $\mathbb{A}$ 2-supernilpotent: $[\mathbb{1}, \mathbb{1}, \mathbb{1}] = \mathbb{0}$.
- When $[\cdot, \cdot]$ is not symmetric, there are many different notions of nilpotence. All of them imply solvability.
- Make $\mathbb{A}$ not solvable.

What does $[\mathbb{1}, \mathbb{1}, \mathbb{1}] = \mathbb{0}$ mean?

$$\mathrm{M}(\mathbb{1}, \mathbb{1}, \mathbb{1}) := \mathrm{Sg}_{\mathbb{A}^{2^3}} \left\{ \begin{tikzcd}[row sep=small, column sep=small] a \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & b \arrow[dr, dash] \arrow[dd, dotted, dash] & \\ & a \arrow[rr, dash] \arrow[dd, dash] & & b \arrow[dd, dash] \\ a \arrow[rr, dotted, dash] \arrow[dr, dash] & & b \arrow[dr, dotted, dash] & \\ & a \arrow[rr, dash] & & b \end{tikzcd}, \begin{tikzcd}[row sep=small, column sep=small] a \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & a \arrow[dr, dash] \arrow[dd, dotted, dash] & \\ & a \arrow[rr, dash] \arrow[dd, dash] & & a \arrow[dd, dash] \\ b \arrow[rr, dotted, dash] \arrow[dr, dash] & & b \arrow[dr, dotted, dash] & \\ & b \arrow[rr, dash] & & b \end{tikzcd}, \begin{tikzcd}[row sep=small, column sep=small] a \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & a \arrow[dr, dash] \arrow[dd, dotted, dash] & \\ & b \arrow[rr, dash] \arrow[dd, dash] & & b \arrow[dd, dash] \\ a \arrow[rr, dotted, dash] \arrow[dr, dash] & & a \arrow[dr, dotted, dash] & \\ & b \arrow[rr, dash] & & b \end{tikzcd} \;\middle|\; a, b \in A \right\}$$

$$\begin{tikzcd}[row sep=small, column sep=small] a \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & {\color{blue}b} \arrow[dr, dash] \arrow[dd, dotted, dash] & \\ & a \arrow[rr, dash] \arrow[dd, dash] & & {\color{blue}f} \arrow[dd, dash] \\ c \arrow[rr, dotted, dash] \arrow[dr, dash] & & d \arrow[dr, dotted, dash] & \\ & c \arrow[rr, dash] & & d \end{tikzcd} \in \mathrm{M}(\mathbb{1}, \mathbb{1}, \mathbb{1}) \qquad \Longrightarrow \qquad {\color{blue} b = f}$$

We will need to carefully analyze the generation of $\mathrm{M}(\mathbb{1}, \mathbb{1}, \mathbb{1})$.

$$\mathrm{M}(\mathbb{1},\mathbb{1},\mathbb{1}) := \mathrm{Sg}_{\mathbb{A}^{2^3}} \left\{ \begin{array}{cccc} a & & b & \\ & a & & b \\ a & & b & \\ & a & & b \end{array}, \begin{array}{cccc} a & & a & \\ & a & & a \\ b & & b & \\ & b & & b \end{array}, \begin{array}{cccc} a & & a & \\ & b & & b \\ a & & a & \\ & b & & b \end{array} \;\middle|\; a, b \in A \right\}$$

$$[\mathbb{1},\mathbb{1},\mathbb{1}] = \mathbb{0} \qquad \text{iff} \qquad \left[\begin{array}{cccc} a & & {\color{blue}b} & \\ & a & & {\color{blue}f} \\ c & & d & \\ & c & & d \end{array} \in \mathrm{M}(\mathbb{1},\mathbb{1},\mathbb{1}) \quad \Longrightarrow \quad {\color{blue}b = f} \right] \qquad (\dagger)$$

What is the simplest way to ensure $[\mathbb{1},\mathbb{1},\mathbb{1}] = \mathbb{0}$?

- consider $\mathbb{A} = \langle A\,;\; t(x,y) \rangle$ where $t(x,y)$ is injective (infinite $\mathbb{A}$)
- generating $\mathrm{M}(\mathbb{1},\mathbb{1},\mathbb{1})$ in layers, look for first failure of $(\dagger)$

$$t\left(\begin{array}{cccc} a_1 & & {\color{blue}b_1} & \\ & a_1 & & {\color{blue}f_1} \\ c_1 & & d_1 & \\ & c_1 & & d_1 \end{array}, \begin{array}{cccc} a_2 & & {\color{blue}b_2} & \\ & a_2 & & {\color{blue}f_2} \\ c_2 & & d_2 & \\ & c_2 & & d_2 \end{array} \right) = {\color{blue}\begin{array}{cccc} a & & b & \\ & a & & f \\ c & & d & \\ & c & & d \end{array}} \qquad (\text{all in } \mathrm{M}(\mathbb{1},\mathbb{1},\mathbb{1}))$$

- $\mathbb{A}$ is 2-supernilpotent! $\mathbb{A}$ **is also nilpotent!**

$$\mathrm{M}(\mathbb{1},\mathbb{1},\mathbb{1}) := \mathrm{Sg}_{\mathbb{A}^{2^3}} \left\{ \begin{array}{cccc} a & & b & \\ & a & & b \\ a & & b & \\ & a & & b \end{array},\ \begin{array}{cccc} a & & a & \\ & a & & a \\ b & & b & \\ & b & & b \end{array},\ \begin{array}{cccc} a & & a & \\ & b & & b \\ a & & a & \\ & b & & b \end{array} \ \middle|\ a, b \in A \right\}$$

$$[\mathbb{1},\mathbb{1},\mathbb{1}] = \mathbb{0} \qquad \text{iff} \qquad \left[\begin{array}{cccc} a & & b & \\ & a & & f \\ c & & d & \\ & c & & d \end{array} \in \mathrm{M}(\mathbb{1},\mathbb{1},\mathbb{1}) \quad \Longrightarrow \quad b = f \right] \qquad (\dagger)$$

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- $\mathbb{A}$ is 2-supernilpotent! $\mathbb{A}$ **is also nilpotent!**

Idea:

- $\mathbb{A} = \langle A\ ;\ t(x,y)\rangle$, injective $t(x,y)$
- carefully redefine $t(x,y)$ to sabotage injectivity and nilpotence
- be clever so that $t(x,y)$ is injective enough to ensure $[\mathbb{1},\mathbb{1},\mathbb{1}] = \mathbb{0}$

Define

$$O \coloneqq \left\{ o_i^j \mid i,j \in \mathbb{N} \right\}, \qquad R \coloneqq \left\{ r_i^j \mid i,j \in \mathbb{N} \right\}, \qquad A \coloneqq O \cup R \cup \mathbb{N}.$$

Fix an injection $s : A^2 \to \mathbb{N}$. Define

$$t\left(\begin{array}{ccc} r^j_{4i} & — & r^j_{4i+2} \\ | & & | \\ r^j_{4i} & — & r^j_{4i+2} \end{array} , \begin{array}{ccc} r^j_{4i} & — & r^j_{4i} \\ | & & | \\ r^j_{4i+2} & - & r^j_{4i+2} \end{array} \right) \coloneqq \begin{array}{ccc} o^j_i & — & r^{j+1}_i \\ | & & | \\ o^j_i & — & r^{j+1}_{i+1} \end{array}$$

Otherwise, define $t(x,y) \coloneqq s(x,y)$. Let $\mathbb{A} = \langle A\ ;\ t(x,y)\rangle$.

Observe: $t(x,y)$ is injective except when the output is in O.

Define $A = O \cup R \cup \mathbb{N}$ and

$$t\left(\begin{array}{ccc} r^j_{4i} & - & r^j_{4i+2} \\ | & & | \\ r^j_{4i} & - & r^j_{4i+2} \end{array}, \begin{array}{ccc} r^j_{4i} & - & r^j_{4i} \\ | & & | \\ r^j_{4i+2} & - & r^j_{4i+2} \end{array}\right) := \begin{array}{ccc} o^j_i & - & r^{j+1}_i \\ | & & | \\ o^j_i & - & r^{j+1}_{i+1} \end{array},$$

otherwise define $t(x,y) := s(x,y)$ for some injection $s : A^2 \to \mathbb{N}$.

$\mathbb{A}$ is not solvable

- Dervived series: $[\alpha]_0 := \alpha,$ $[\alpha]_{n+1} := \big[[\alpha]_n, [\alpha]_n\big].$
- $R^j := \{r^j_i \mid i, j \in \mathbb{N}\}$ is contained in a $[\mathbb{1}]_j$-class (induction on j).
- Thus $[\mathbb{1}]_j \neq \mathbb{0}$ for any j, so $\mathbb{A}$ is not solvable.

Define $A = O \cup R \cup \mathbb{N}$ and

$$t\left(\begin{array}{ccc} r^j_{4i} & - & r^j_{4i+2} \\ | & & | \\ r^j_{4i} & - & r^j_{4i+2} \end{array}, \begin{array}{ccc} r^j_{4i} & — & r^j_{4i} \\ | & & | \\ r^j_{4i+2} & \cdot & r^j_{4i+2} \end{array}\right) := \begin{array}{ccc} o^j_i & - & r^{j+1}_i \\ | & & | \\ o^j_i & - & r^{j+1}_{i+1} \end{array},$$

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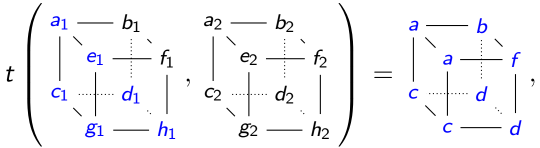
Generating $\mathrm{M}(\mathbb{1}, \mathbb{1}, \mathbb{1})$ in layers, look for first instance of



$b \neq f$

Observations

- If $t(a, b) = t(c, d)$ then $a = c$.
- Argument squares: if 3 back-to-front edges are equal, then the 4th is too.
- $b_2 \neq f_2$, otherwise $b = f$. At most 2 back-to-front edges in the second argument cube are equal.

The rest of the argument is technical ...

Define $A = O \cup R \cup \mathbb{N}$ and

$$t\left(\begin{array}{ccc} r^j_{4i} & - & r^j_{4i+2} \\ | & & | \\ r^j_{4i} & - & r^j_{4i+2} \end{array}, \begin{array}{ccc} r^j_{4i} & — & r^j_{4i} \\ | & & | \\ r^j_{4i+2} & \cdot & r^j_{4i+2} \end{array}\right) := \begin{array}{ccc} o^j_i & — & r^{j+1}_i \\ | & & | \\ o^j_i & — & r^{j+1}_{i+1} \end{array},$$

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Generating $\mathrm{M}(\mathbb{1}, \mathbb{1}, \mathbb{1})$ in layers, look for first instance of

$$t\left(\begin{array}{cccc} a_1 & & {\color{blue}b_1} & \\ & a_1 & & {\color{blue}b_1} \\ c_1 & & d_1 & \\ & c_1 & & d_1 \end{array}, \begin{array}{cccc} a_2 & & {\color{blue}b_2} & \\ & e_2 & & {\color{blue}f_2} \\ c_2 & & d_2 & \\ & g_2 & & h_2 \end{array}\right) = \begin{array}{cccc} {\color{blue}a} & & {\color{blue}b} & \\ & {\color{blue}a} & & {\color{blue}f} \\ {\color{blue}c} & & {\color{blue}d} & \\ & {\color{blue}c} & & {\color{blue}d} \end{array}, \qquad b \neq f$$

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Theorem (M+M)

The algebra $\mathbb{A}$ is 3-supernilpotent but not solvable.

- nilpotence and solvability can be generalized using the *n*-ary commutator
- call the generalized notions *n*-dimensional nilpotence and solvability
- a similar construction yields $\mathbb{A}_n := \langle A\,;\ t(x_1, \ldots, x_n)\rangle$.

Theorem (M+M)

The algebra $\mathbb{A}_n$ is $(n+1)$-supernilpotent but not n-dimensional solvable.

Supernilpotence Need Not Imply Nilpotence

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Thank you for your attention.